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Math. 426

Math. 426

A
COMPLEAT SYSTEM
OF
OPTICKS

In Four BOOKS, viz.

A Popular, a Mathematical, a Mechanical, and
a Philoſophical Treatiſe.

To which are added
REMARKS upon the Whole.

BY

ROBERT SMITH LL.D.

Profeſſor of Aſtronomy and Experimental Philoſophy at CAMBRIDGE,
and Maſter of Mechanicks to his MAJESTY.

> *Quid tam mirabile, quam particulam corporis quandam ita fabricatam eſſe, ut ejus opera animal ſentiat procul poſitorum corporum figuram, poſitum, motum quemlibet, diſtantiam; idque etiam cum colorum varietate, quo diſtinctius ea dignoſceret? Nihil eſt, in quo manifeſtius Geometriæ artem Deus exercuerit.* Hugenii Coſmotheoros. p. 40.

CAMBRIDGE,
Printed for the AUTHOR, and ſold there by *Cornelius Crownfield*, and at *London*
by *Stephen Auſten* at the *Angel* and *Bible* in *St. Paul's Church-yard*, and
Robert Dodſley at *Tully's Head* in *Pall-Mall.* MDCCXXXVIII.

TO

THE RIGHT HONOURABLE

EDWARD WALPOLE Eſq;

Principal Secretary to his GRACE the DUKE of DEVONSHIRE as Lord-Lieutenant of IRELAND, and one of His MAJESTY's moſt Honourable Privy-Council for that Kingdom.

SIR,

OF all the incidents in my life there is none I reflect upon with greater pleaſure, than the happineſs of falling into your acquaintance at CAMBRIDGE. There, among the many agreeable hours I had the honour of paſſing in your company, you may remember the converſation ſometimes turned upon Opticks, a Science you knew I had taken ſome pains in: And the curioſity you then diſcovered, in relation to ſome points that had not been ſufficiently explained by former writers, was the principal motive with me to take them more particularly into conſideration.

BUT as our acquaintance grew by degrees into a ſettled friendſhip, I became deſirous of giving you ſome publick teſtimony of the reſpect and affection I had for you; to which end I began to collect together and reviſe my ſcattered papers, and to think of preparing them for the Preſs. And ſo ſollicitous was I to produce ſomething not unworthy of the Patronage you were then pleaſed to grant me, that I attended too little to the good old rule, *Quid valeant humeri*, and projected a deſign much too large for the health and leiſure I had to proſecute it.

BY the want of theſe, the work has been retarded for ſo many years, that in the more active and conſpicuous ſcene of life you have ſince entered upon, wherein you appear with ſo much honour to yourſelf and ſo much pleaſure to your friends, when your former thoughts upon this ſubject muſt neceſſarily have been long ſince diſcontinued and laid aſide, I am ſenſible you can hardly have time to look into, or think of what in the days of your Academical leiſure would have been no diſagreeable entertainment.

To you however I muſt beg leave to addreſs it, becauſe, of how little uſe or pleaſure ſoever it may now be to you, to you the Preſent of right belongs. 'Twas you that ſet me firſt to work; 'twas you that procured me leiſure in time to finiſh it; 'twas my affection for you and that alone, that gave me ſtrength and ſpirit to go through with it.

For in the weak and unſettled ſtate of health I have been in for many years, the common motives to ſuch undertakings, as the importunity of acquaintance, the hopes of fame, or even the deſire of promoting uſeful knowledge, could never have been ſtrong enough to carry me through ſo large and difficult a Work, had I not been animated by a warmer affection ariſing from the amiable qualities of the beſt of Friends, improved by a long experience and obſervation of them, and ſtill farther heightened by a train of favours and honours received from his hand and heart.

THESE effects of your Friendſhip for me are ſo much to my reputation, that you muſt pardon my vanity in taking this publick notice of them. Were the Treatiſe I preſent you with at all anſwerable to my ambition, it ſhould be a laſting Monument of the Gratitude of,

SIR,

Your moſt affectionate Friend,

and humble Servant,

ROBERT SMITH.

PREFACE.

THE firſt of the four Books, or the Popular Treatiſe, though partly intended as an Introduction to the reſt, is chiefly deſigned for the uſe of thoſe who would know ſomething of Opticks, but want the preparatory learning that is neceſſary for a thorough acquaintance with that Science. With this view I have here avoided all Geometrical Demonſtrations, and inſtead thereof have ſubſtituted that more entertaining and looſer ſort of proof, that may be drawn from experiment only; and the experiments I have contrived for that end, are not only eaſy to be underſtood, but may, if the Reader pleaſes, be tryed with very little trouble or apparatus

By this means one who uſes a little application, may find himſelf maſter of no inconſiderable part of the doctrine of Opticks, which is here explained in ſuch a manner, as I hope may be eaſy to all, and yet not tedious to more ſkilful Readers, who perhaps may find here and there ſomething not unworthy of their notice.

Another advantage deſigned for the Readers above-mentioned is, that this Popular Treatiſe well underſtood, will be abundantly ſufficient to conduct them through many curious pieces contained in the Remarks, and even through the whole third and fourth Books; eſpecially if their heads be a little turned towards mechanical matters, and be furniſhed with ſome of the firſt and eaſieſt principles of Aſtronomy.

In the third Book, beſides ſome curious collections belonging to the art of grinding Glaſſes, drawn up and communicated by the late Hon. Samuel Molyneux *Eſq; I have given a full deſcription of a compleat Set of Optical and Aſtronomical Inſtruments, according to the lateſt and beſt improvements; together with particular explanations of their ſeveral uſes, when applyed to the purpoſes of Aſtronomy, Geography, Navigation, Levelling, and other uſeful Arts.*

And in the fourth Book I have given a compleat hiſtory and collection of Teleſcopical Diſcoveries in the heavens, ſelected from a great variety of books, memoirs, and obſervations of the beſt Aſtronomers. To which I intended to have added a like collection of Microſcopical Diſcoveries, but found my Work was grown too large without it.

Thus I have endeavoured to lay open a large field of Knowledge, even to perſons very moderately prepared for this ſort of Learning.

Thoſe that are better verſed in Euclid's *Elements, after reading the four firſt chapters of my ſecond Book, which contain the Geometrical Elements of the whole Work, demonſtrated in a ſhort and eaſy manner, may then proceed to ſuch parts of it as they are moſt inclined to. For I have endeavoured to make all the Chapters in the four Books, and the pieces in the Remarks, as much independent on one another as I could; chuſing rather to repeat a few matters in the order and form in which they occaſionally preſented themſelves, than to trouble my readers to recur to them elſewhere in a different dreſs: not to mention that different views of the ſame truths are ſeldom diſagreeable to men of taſte, and are equally uſeful to beginners with the writings of different authors upon the ſame ſubject.*

As I generally demonſtrate every thing from the firſt principles, the Remarks are ſeldom explanatory of any thing in the four Books, but are chiefly additions to them; as containing the Hiſtory of Inventions and their improvements, diſſertations on different opinions in difficult points, confutations of errors, collections of hints, obſervations and queries, ſubjects enlarged upon or reconſidered in a different manner, both phyſical and mathematical, as curious and valuable as any in the four Books: and as ſome of theſe pieces are pretty long, I ſhould have printed them with a letter ſomewhat larger, had the preſs been furniſhed with it.

As the general contents of the whole Work will ſufficiently appear by the Table of the Chapters, and the particulars by the Index, I might here conclude; but as the Reader may expect ſome account of what is to be found in this Work, either of new diſcovery, or of improvement upon what has been delivered by other writers, I ſhall here mention ſome of the principal matters.

1. *The determination of the Focus of a pencil of reflected rays, after falling directly or obliquely on any number of plane or ſpherical ſurfaces, or of refracted rays, after paſſing directly or obliquely through any number of lens's of any thickneſs, or through different mediums having plane or ſpherical ſurfaces, is made more general and eaſy; by reducing it in all theſe caſes to the like ſimple proportion, that determines the focus of a pencil after falling directly upon a ſingle ſurface: and even in the Popular Treatiſe, I have given a plainer and fuller idea of the poſitions and relative motions of conjugate focus's along the axis of the glaſſes, than what I could meet with in Books of Opticks.*

2. *The determination of the Aberrations of reflected and refracted rays from the geometrical focus, cauſed by their different refrangibility and by the ſphericalneſs of the figure of the ſurfaces, is made eaſier in ſimple caſes, is treated more copiouſly in different ways, is made more general, and applied to more complex conſtructions of Optical Inſtruments than heretofore.*

3. *Hence after demonſtrating the known rules for proportioning the lengths, apertures and eye-glaſſes of reflecting and refracting Teleſcopes, the theory of a reflecting Microſcope, having a concave ſpherical ſpeculum and a convex eye-glaſs, as propoſed by Sir* Iſaac Newton, *is fully conſidered; and rules are given for the improvement of this microſcope, as far as its conſtruction will admit.*

4. *The rules delivered by Mr.* Huygens *for the improvement of refracting Microſcopes, both ſingle and double, are alſo conſidered and demonſtrated But ſince the magnifying powers of all ſorts of microſcopes yet extant, are limited by the inſuperable difficulty of truly figuring a lens or ſpeculum of ſo ſmall a ſize, as their conſtructions require for magnifying more than ordinary; in conſidering how to remove that difficulty, I found out a conſtruction with two ſpherical ſpeculums and a convex eye-glaſs, wherein that exceſſive ſmallneſs is not neceſſary. And after due conſideration of the theory here deſcribed, and an Eſſay of it made by Mr.* Short *of* Edinburgh, *I am of opinion, if ſufficient care be taken in the practical part, to follow the directions and the Table of Meaſures here given, that this microſcope may far excel all others yet extant: I mean in magnifying tranſparent objects more than uſual; which is generally ſufficient, becauſe the particles of all ſubſtances will become tranſparent when rendred ſufficiently ſmall, and theſe are the particles that we want to ſee magnified more than ordinary.*

5. *Mr.* Gregorie *propoſed to conſtruct his Teleſcope with ſpeculums figured according to the conick ſections, which being impracticable, the neceſſity of uſing ſpherical ſpeculums has rendred the theory much more complex, on account of the aberrations of the rays. His teleſcope and that of Mr.* Caſſegrain *I have conſidered very minutely, and have given a ſolution of this problem. Having the focal diſtance of the larger ſpeculum, the angle of viſion, the degrees of apparent brightneſs and diſtinctneſs, with which the object ſhall appear, to conſtruct the teleſcope. Hence I have calculated a table of the dimenſions and magnifying powers of theſe teleſcopes.*

6. *To the deſcription of the Binocular Teleſcope I have added a ſolution of the moſt remarkable phænomenon belonging to it, and of two or three more of the ſame kind relating to double viſion.*

7. *To the deſcription and properties of Cauſticks, I have added in ſome caſes a determination of the denſity of the rays in their ſeveral parts; and have compared the powers of Burning-Glaſſes of ſeveral ſorts, one with another; whereby it appears which glaſſes are the beſt for caſting a ſtrong light upon microſcopical objects.*

8. *I have given a more general ſolution of the known problem for finding the diameters and breadths of Rain-bows; and have compoſed ſome propoſitions to ſhew the variations of the apparent magnitude, figure and brightneſs of the Sun, when ſeen by rays refracted in various angles through ſpherical bodies; and have conſidered Sir* Iſaac Newton*'s thoughts upon Halo's, whoſe phænomena, he tells us in the preface to his Opticks, he endeavoured to account for, but for want of ſufficient obſervations, left that matter to be farther examined. I have alſo reduced the mathematical matters belonging to Mr.* Huygens*'s theory of Corona's and Parhelia, to a few propoſitions, and demonſtrated the conſtruction of his tables. And have ſhewn why Corona's or Halo's about the ſun and moon, appear not circular but oval.*

9. *This is a natural conſequence from our idea of the Sky, whoſe apparent Figure I have here conſidered; and taking it for a ſegment of a ſpherical ſurface, as it generally ſeems to be, have determined the proportion of its altitude to the diameter of its baſe; and from hence have deduced an adequate ſolution of the long diſputed queſtion, why the Sun, Moon and Conſtellations appear larger near the horizon, than at higher elevations, and in what proportions; which proportions agree ſo well to our common conceptions of their different magnitudes at different elevations, as to amount to a phyſical proof of the truth of this ſolution; eſpecially as it is applicable to many other phænomena of the ſame kind, and is confirmed in the Remarks by an experiment made upon a like appearance. I have alſo offered ſome reaſons, to be farther examined, why the Horizontal Moon appears now and then of a ſize extraordinary large; and have determined the proportion of moon-light to day-light.*

10. *The cauſes that ſuggeſt our ideas of Diſtance, and the determination of the Apparent Diſtance of an object ſeen in glaſſes, is another famous inquiry of no ſmall difficulty, upon which much has been written, but with little certainty and ſatisfaction to the curious. I have therefore conſidered this point in*

in a very particular manner, and have ſettled it on ſuch a foundation of reaſon and experience, as, I hope, will admit of no doubt or diſpute for the future. And upon the Principle by which I introduce the conſideration of apparent diſtance into geometry, I have not only determined it in viſion with any number of glaſſes, but by the help of Geometrical Places, have ſhewn its regular variations, while the eye, object, or ſyſtem of glaſſes are moving forwards or backwards; and have found the variations ſo determined to be agreeable to experience.

11. By the help of the ſaid Principle, and of an admirable Dioptrick Theorem invented by Mr. Cotes, I was alſo enabled to give very general and yet very eaſy determinations of the Apparent Diſtance, Magnitude, Situation, Diſtinctneſs, Brightneſs, the greateſt Angle of Viſion and Viſible Area, that is, of all the appearances of an object ſeen by rays coming from any number of ſpeculums, lens's or mediums having plane or ſpherical ſurfaces; and in corollaries from them to deduce the known properties of Teleſcopes and Microſcopes of all ſorts; which however are independently demonſtrated in other places of the Book and Remarks.

12. In farther confirmation of the truth and extent of the Principle abovementioned, I have alſo applied it to one of the moſt difficult ſubjects in Opticks, upon which the beſt writers have not yet ſucceeded. It is to determine the Apparent Shape of a large plane object, diſtorted by too oblique reflections from ſpherical ſpeculums, or too great refractions through ſpherical mediums, when viewed either by one eye alone or both; which in ſome caſes alters the appearance very ſurprizingly, and by the bye accounts for that admirable effect of a large concave ſpeculum in heightening the Relievo of Pictures.

13. Laſtly, I have drawn up ſome general Theorems on purpoſe for computing the diameter of the Image of an object, whether diſtinctly or indiſtinctly formed upon the Retina or any ſurface parallel to it; and for ſhewing its properties and variations upon varying the diſtance of the object; alſo for computing the diameter, and ſhewing the variations of the ſection of a ſingle pencil cut by the retina or a ſurface parallel to it; and for determining the place of one or more refracting ſurfaces, requiſite to transfer the rays from one given focus to another.

This is a ſhort account of the general matters to be found in this Treatiſe, beſides collections from other authors, whoſe thoughts and words too I have not ſcrupled to copy for many pages together, as often as they ſuited my purpoſe; and when they did not, thoſe that are converſant in Books of Opticks will find, *that what I have borrowed from others, is not the worſe for having paſſed through my hands.*

I muſt now make my acknowledgements to my Friends and Acquaintance for the aſſiſtance they have been pleaſed to give me; viz. *to* Martin Folkes *Eſq*; *for his curious remarks on fallacies in viſion, on the ſun's apparent diſtance, on the apparent figure of the ſky, on the apparent curvity of the ſides of long walks and ploughed lands, and the changes of curvity by the obſerver's motion*; *to* John Hadley *Eſq*; *to whom we owe the preſent perfection of reflecting teleſcopes, for a full and accurate deſcription of his manner of making the ſpeculums*; *to Mr.* Mac Laurin *Profeſſor of Mathematicks at* Edinburgh, *for his account of ſome reflecting teleſcopes made with glaſs ſpeculums by Mr.* Short *of* Edinburgh; *to Mr.* George Graham *F. R. S. for aſſiſting me in the deſcription of his Aſtronomical Sector, and of the Mural Arch in the Royal Obſervatory at* Greenwich; *and to Dr.* Jurin, *for his ſolution of Mr.* Molyneux's *problem mentioned by Mr.* Locke, *whether a perſon born blind and made to ſee when adult, could diſtinguiſh a globe from a cube at firſt ſight*; *for his remarks on fallacies of ſight, and on the aſſociation of ideas*; *for his diſſertation on ſquinting*; *for his experiments to ſhew how much brighter an object appears to both eyes than to one alone*; *for his ſolution of ſome ſurprizing phænomena in double viſion*; *and laſtly for his* Eſſay on diſtinct and indiſtinct viſion; *which contains ſo great a variety of new obſervations, curious diſcoveries, and difficult points, diſcuſſed and determined with ſo much perſpicuity, penetration and judgment, that for a juſt idea of them the reader muſt have recourſe to the Eſſay it ſelf. When he has read it and the curious pieces communicated by thoſe other gentlemen, I am ſure he will join with me in thanks to their ſeveral Authors.*

ERRATUM.

Essay pag. 149. Col. 1. lin. 10. *from the bottom, read,* as the image of the diameter of the ſilk.

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GEORGE R.

GEORGE the Second, by the Grace of God, King of *Great Britain*, *France* and *Ireland*, Defender of the Faith *&c.* To All, to whom theſe Preſents ſhall come, Greeting. Whereas our Truſty and Well-beloved ROBERT SMITH, Doctor of Laws, Profeſſor of Aſtronomy and Experimental Philoſophy in Our Univerſity of *Cambridge*, has humbly repreſented unto Us, that he has with great Labour and Expence compoſed, and prepared for Publication, ſeveral Philoſophical, Optical, Mathematical, and Aſtronomical Works, to the Copies of which he hath the ſole Right and Title; and hath humbly beſought Us, to grant him Our Royal Privilege and Licence for the ſole Printing and Publiſhing them, for the Term of Fourteen Years; We, being willing to give due Encouragement to Works of this Nature, that tend to the Advancement and Propagation of Learning, are graciouſly pleaſed to condeſcend to his Requeſt, and do, therefore by theſe Preſents, ſo far as may be agreeable to the Statute in that Behalf made and provided, grant unto him the ſaid ROBERT SMITH, his Heirs, Executors and Aſſigns, Our Royal Privilege and Licence, for the ſole Printing and Publiſhing the ſaid Works, for the Term of Fourteen Years, to be computed from the Date hereof; Strictly forbidding and prohibiting all Our Subjects within our Kingdoms and Dominions to Reprint, Abridge or Tranſlate the ſame, either in the like or any other Volume or Volumes whatſoever; or to import, buy, vend, utter or diſtribute any Copies of them, reprinted beyond the Seas, during the ſaid Term of Fourteen Years, without the Conſent and Approbation of the ſaid ROBERT SMITH, his Heirs, Executors, and Aſſigns, by Writing under his, or their Hands and Seals firſt had and obtained, as they, and every of them offending herein, will anſwer the contrary at their Perils; Whereof the Commiſſioners and other Officers of Our Cuſtoms, the Maſter, Wardens and Company of Stationers of Our City of *London*, and all Others Our Officers and Miniſters, whom it may concern, are to take Notice, that due Obedience be given to Our Pleaſure herein ſignified. Given at Our Court at St. *James's* the Seventh Day of November One Thouſand Seven Hundred and Thirty Seven. In the Eleventh Year of Our Reign.

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BOOK I

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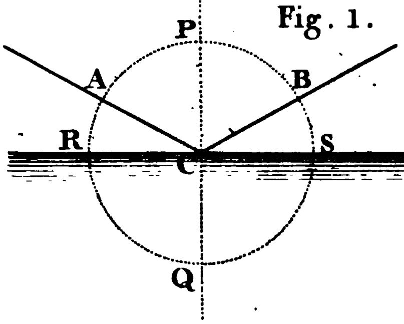

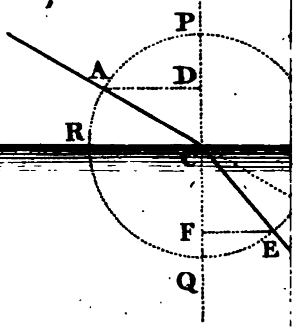

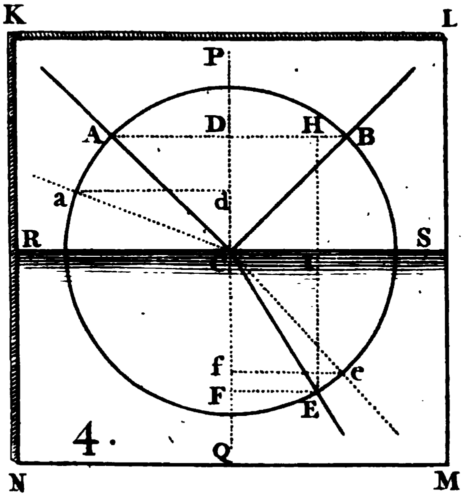

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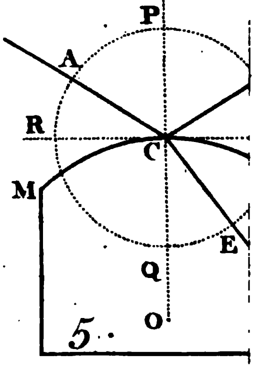

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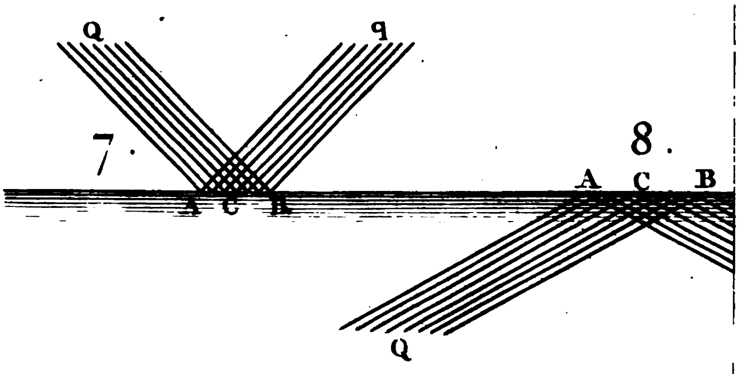

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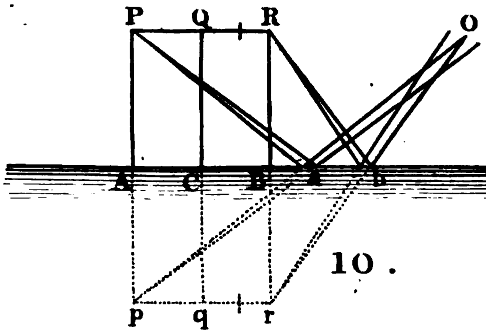

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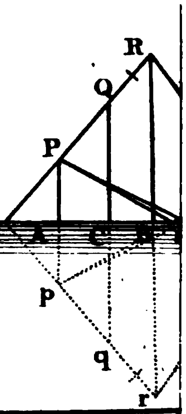

A
COMPLEAT SYSTEM
OF
OPTICKS.

BOOK I.

A
Popular Treatiſe.

CHAP. I.

CONCERNING LIGHT.

Light conſiſts of parts.

WHOEVER has conſidered what a number of properties and effects of light are exactly ſimilar to the properties and effects of bodies of a ſenſible bulk, will find it difficult to conceive that light is any thing elſe but very ſmall and diſtinct particles of matter[*]: which being inceſſantly thrown out from ſhining ſubſtances, and every way diſperſed by reflection from all others, do impreſs upon our organs of ſeeing that peculiar motion, which is requiſite to excite in our minds the ſenſation of light. But for the preſent purpoſe it is ſufficient to obſerve that light conſiſts of parts, both ſucceſſive in the ſame lines and contemporary in ſeveral lines: becauſe in the ſame place, you may ſtop that which comes one moment, and let paſs that which comes preſently after; and at the ſame time, you may ſtop it in one place, and let it paſs in another. For that part of the light which is ſtopt cannot be the ſame with that which is let paſs.

A ray of light what and how conſidered.

2. The leaſt light or part of light, which may be ſtopt alone without the reſt of the light, or propagated alone, or do or ſuffer any thing alone, which the reſt of the light doth not or ſuffereth not, is called a Ray of light[†]. That rays of light are ſtraight, is evident enough from the ſhadows of bodies; or from the appearance

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[*] Newt. Opt. Qu. 29. p. 345. 8°. Edit.

[†] Newton's definition. Opt. p. 1.

of light paſſing through little holes into a dark room full of duſt or ſmoke; or becauſe bodies cannot be ſeen through the bore of a bended pipe; or becauſe they ceaſe to be ſeen by the interpoſition of other bodies, as the fixt ſtars by the interpoſition of the moon and planets; and the parts of the ſun by the interpoſition of the Moon, Mercury or Venus. Rays of light may therefore be repreſented by ſtraight lines, not Mathematical but Phyſical, which are deſcribed by the motion of the parts or particles of light: and the point which a ray poſſeſſes in falling upon any ſurface may be conſidered as a Phyſical Point.

The manner of reflection and refraction of a ray deſcribed. Fig. 1.

3. When a ray of light falls obliquely upon a ſmooth poliſhed ſurface, it is turned out of its way either by reflection or refraction in the following manner. Imagine the paper upon which this figure is drawn to be perpendicular to the ſurface of ſtagnating water, and to cut it in the line *RS*, and that a ray of light, coming in the air along the line *AC*, falls upon *RS* at the point *C*. Then ſuppoſing the line *PCQ* to be perpendicular to the ſurface of the water, if the ray be reflected, or turned back at *C* into the air again, it will deſcribe a ſtraight line *CB*, inclined to the perpendicular *CP* at an angle *PCB* exactly equal to the angle *PCA*.

Fig. 2.

But if the ray that came along *AC* goes into the water at *C*, it will not proceed ſtraight forward, but being refracted or bent at *C*, it will deſcribe another ſtraight line *CE* inclined to the perpendicular *CQ* at a leſſer angle *ECQ* than the angle *ACP*; and the line *CE* will always be ſo ſituated, that when any circle, deſcribed about the center *C*, cuts the line *CA* in *A* and *CE* in *E*, the perpendiculars *AD* and *EF*, drawn from *A* and *E* to the line *PQ*, ſhall always bear the ſame proportion to each other; whatever be the magnitude of the angle *ACP*. In water the line *EF* is always three quarters of *AD*.

Angles and ſines of incidence and refraction what. Fig. 1, 2.

4. In both theſe caſes the line *AC* is called the Incident Ray, *CB* the Reflected Ray, *CE* the Refracted Ray, *C* the point of incidence, *PCQ* the perpendicular (at the point) of incidence, the angle *ACP* the Angle of Incidence, *BCP* the Angle of Reflection, *ECQ* the Angle of Refraction; the line *AD* the Sine of Incidence, that is, of the angle of incidence; and *EF* the Sine of Refraction, that is, of the angle of refraction.

A Medium what.

5. Empty ſpace, or any tranſparent body, is called a Medium; and mediums are denſer in proportion as they are heavier bulk for bulk; and their power to reflect and refract light is found to be greater in proportion as they are denſer, very nearly *.

Laws of reflection and refraction what.

6. The foregoing properties of Reflection and Refraction being diſcovered and eſtabliſhed by repeated experiments upon light and bodies

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* Newt. Opt. p. 245. 8°.

of all ſorts both fluid and ſolid, without any exception yet known; and being the principal foundation of the whole ſcience of Opticks, are called the Laws of Reflection and Refraction; and are expreſſed by Sir *Iſaac Newton* in the following words.

7. *The angles of reflection and refraction lye in one and the ſame plane with the angle of incidence*; that is, in the plane drawn through the incident ray and the perpendicular at the point of incidence, as repreſented in the figures, 1, 2. First law.

8. *The angle of reflection is equal to the angle of incidence.* Second law.

9. Hence it follows that the incident and reflected rays are equally inclined to the reflecting plane; that is, the angles ACR and BCS are equal; as appears by taking the equal angles PCA and PCB from the equal angles PCR and PCS. First conſequence. Fig. 1.

10. It follows alſo that when the incident ray is perpendicular to the reflecting ſurface, it ſhall be reflected directly back along the ſame perpendicular; as appears by diminiſhing the equal angles of incidence and reflection till the rays AC, CB coincide with the perpendicular CP. Second conſequence.

11. *If the reflected or refracted ray be returned directly back to the point of incidence, it ſhall be reflected or refracted into the ſame line before deſcribed by the incident ray.* Third law.

12. *Refraction out of a rarer medium into a denſer*[a] *is made towards the perpendicular; that is, ſo that the angle of refraction be leſs than the angle of incidence.* Fourth law.

13. *The ſine of incidence, AD, is to the ſine of refraction, EF, either accurately or very nearly in a given ratio*; that is, ſuppoſing any other incident ray aC to be refracted into the line Ce, and the ſines ad and ef to be drawn perpendicular to PQ, the ratio of ad to ef is the ſame as the ratio of AD to EF. It is found by experience, that if the refraction be made out of air into water, the ſine of incidence of red light is to the ſine of its refraction as 4 to 3: if out of air into glaſs as 17 to 11, or nearly as 3 to 2. In light of other colours the ſines have other proportions, but the difference is ſo little that it ſeldom need be conſidered. Fifth law. Fig. 4.

14. Hence it appears by inſpection of the figures (2, 3, 4.) that when the angle of incidence ACP is increaſed, the correſponding angle of refraction ECQ will alſo be increaſed; becauſe the ratio of their ſines, AD, EF, cannot continue the ſame unleſs they be both increaſed. Conſequently if two angles of incidence be equal to each other, the angles of refraction will alſo be equal to each other. On the contrary, when the angle of incidence is diminiſhed, the angle of refraction will alſo be diminiſhed; inſomuch that when one of theſe angles becomes infinitely ſmall the other alſo becomes infinitely ſmall. First conſequence.

[a] Art. 5.

Second conſequence.

15. And ſo it comes to paſs that when the incident ray coincides with the perpendicular to the refracting ſurface, it will proceed ſtraight forward into the other medium without any bending at all.

Third conſequence.

Fig. 2.

Fig. 3.

[a] Art. 13.

16. From which it is reaſonable enough to conclude back again, that while the angle of incidence is continually increaſing, the refracted ray will be continually more and more bent and diverted from the courſe of the incident ray produced: I mean if AC be continued to G; the arch EG and the angle ECG will continually increaſe: eſpecially conſidering that when the angle of incidence in air becomes very nearly a right one, and conſequently the incident ray goes almoſt parallel to the ſurface of the water, this ray is as much bent at C into the line CE as the 3d figure repreſents. In which EF, the ſine of refraction, being always three quarters [a] of AD, is now three quarters of the radius of the circle. Hence we find * that this angle of refraction, ECQ, is about $48\frac{1}{2}$ degrees: and ſo the angle ECS (being its complement to 90 degrees) is about $41\frac{1}{2}$ degrees; which in this caſe meaſures the deviation of the ray from its firſt courſe along the ſurface of the water. The deviation at the ſurface of glaſs is greater than at the ſurface of water; the ratio of the ſines being greater, that is, as 3 to 2 or nearer as 31 to 20. Hence we find that the angle ECQ is about 40 and ECS about 50 degrees.

Refraction changed into reflection.

17. The bending and deviation is the ſame when the ray goes back again along the ſame lines EC, CA; and if an angle of incidence eCQ be any thing greater than about $48\frac{1}{2}$ degrees in water, or any thing greater than about 40 in glaſs, this ray eC will not be refracted into air, but will be reflected into the line Cf, making the angle of reflection QCf equal to the angle of incidence QCe: as will appear by experiment in the ſixth chapter.

Experimental proof of theſe Laws of reflection and refraction.

Fig. 4.

18. The truth of theſe laws and of all the conſequences drawn from them may be eaſily examined in the manner following. Upon a ſmooth board $KLMN$, about a center C with any opening of the compaſſes (the larger the better) deſcribe a circle $PRQS$; and having drawn two diameters PQ and RS perpendicular to each other, from the point P, with any opening of the compaſſes, cut off equal arches PA, PB, and draw the lines CA, CB; then ſticking three pins perpendicular to the board at the points A, B, C, dip the board into water as far as the line RS; and holding it perpendicular to the ſurface of the water, look along the pins A, C; and an image of the pin B will appear in the water in the line AC produced. Which ſhews that the ray which came from the pin B is reflected from the water, at the point C, along the line CA to the eye of the ſpectator. If the pin at C touches the water, it will diſturb the ſmoothneſs of its ſurface;

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* By a Table of ſines.

and therefore it is better not to place it in the center, but a little higher in the line *CA*. The event will be the ſame if the reflection be made by any other fluid or ſolid body, as may be tryed by cutting off the lower ſemicircle, and by placing the diameter, *RS*, of the upper ſemicircle upon the ſurface of the ſolid.

Upon the ſame board draw the line *AB* cutting *CP* in *D*, and from the lines *DB* and *CS* cut off *DH* and *CI*, each equal to three quarters of *DA*, and through the points *H*, *I*, draw the line *HIE*, cutting the circumference in *E*; and the perpendicular *EF* drawn from *E* upon *PQ*, will be equal to *DH*, or three quarters of *DA*. Then ſtick another pin at *E*, and the board being dipped into water, as before, the pin at *E* will appear to the eye to be in the ſame line with the pins at *A* and *C*. Which ſhews that the ray which comes from the pin *E* is ſo refracted at *C*, as to advance to the eye along the line *CA*; and therefore when the refraction is made out of water into air, *EF* the ſine of incidence, is to *AD* the ſine of refraction, as 3 to 4. If other pins be fixed any where in the line *CE*, they will all appear in the line *AC* produced: and the whole line *CE* will appear in the water as if it were a continuation of *AC* ſtraight forward. Which ſhews that the ray which comes from the pin *E*, deſcribes a ſtraight line in the water; and that it is bent at the ſurface only. On the contrary, if an opportunity be taken when the Sun is juſt ſo high, that the ſhadow of the pin *A* ſhall coincide with the line *AC*, the refracted ſhadow will coincide with the line *CE*. Or whatever be the Sun's height, move the pin *A* higher or lower till the ſhadow falls upon the center *C*, and there fix it, ſuppoſe at *a*; then ſticking the compaſſes into any point of the refracted ſhadow, take up the board, and through this point and the center *C* draw a line *Ce*, cutting the circle in a new point *e*; and the ratio of the new perpendiculars, *ad* and *ef*, will be the ſame as before; that is, as 4 to 3, as near as can be meaſured.

19. Laſtly it is to be obſerved, that a ray of light is reflected or refracted at a ſpherical ſurface according to the ſame laws as if it were reflected or refracted at a plane, touching the ſpherical ſurface at the point of incidence. Let *AC* be a ray of light falling upon any point *C* of a ſpherical ſurface *MCN*, repreſented by the arch *MCN*, whoſe center is *O*; through the points *O* and *C* draw the line *PQ*, and the line *RCS* perpendicular to it, repreſenting a plane ſurface touching the ſpherical ſurface at *C*. Now becauſe a ray of light is conſidered as a phyſical line, and is refracted or reflected at a phyſical point [a], which is common to both ſurfaces, *MCN* and *RCS*, it follows that the refracted or reflected ray will take the ſame courſe in both caſes. And this argument is alſo confirmed by univerſal experience.

This proof applied to ſpherical ſurfaces.

Fig. 5, 6.

[a] Art. 18.

Chap.

CHAP. II.

CONCERNING GLASSES.

Deſign.

HAVING conſidered the reflection and refraction of a ſingle ray at a ſingle ſurface, I proceed to ſhew how one or more pencils or parcels of rays are reflected and refracted, firſt by a ſingle ſurface alone, and then by ſeveral ſurfaces of plane and ſpherical glaſſes; and alſo how images or pictures of objects are formed by them. Though the manner and reaſon of all theſe effects of glaſſes upon light, may be ſufficiently explained by the laws or experiments above mentioned[a] without geometrical propoſitions; yet, I think, the exact quantities of theſe effects cannot be determined by ſo few experiments, without the aſſiſtance of geometry. Therefore to avoid the uſe of it in this firſt book, I have added a few more experiments to the end of this chapter, by which thoſe quantities may be eaſily determined.

An object what and how it radiates.

20. As rays of light are inceſſantly thrown out and diſperſed in all poſſible directions from every point of a luminous body; ſo when they illuminate other bodies, on which they fall, they are alſo inceſſantly thrown back from every point of theſe bodies. For the points of opake bodies ſo enlightened are viſible to the eye at any point of ſpace and in any point of time, as well as the points of the luminous body that enlightened them. The numberleſs rays which flow from all viſible bodies, called objects, may be methodically diſtributed in this manner. The ſurface of the object is conſidered as conſiſting of phyſical lines, and theſe lines as conſiſting of phyſical points, and theſe points are conceived to radiate all manner of ways. It is uſual to make uſe of nothing elſe for an object but a phyſical line. For by how much that line is increaſed or diminiſhed in apparent magnitude or brightneſs or diſtinctneſs, ſo much the diameter or length of any object, in its place, would be increaſed or diminiſhed.

A focus, pencil, parallel rays what
Fig. 9.

21. The point *Q* from which rays diverge, or towards which they converge (being made to go back towards the ſame point though they may never meet at it) is called their focus. And in both caſes any parcel of theſe rays, as *QBC* or *QBA*, conſidered apart from the reſt, is called a pencil of rays; and theſe rays are ſaid to belong to that focus, whether it be near at hand or at an immenſe diſtance; and in the latter caſe the rays are called, and conſidered as, parallel or equidiſtant from each other; becauſe the difference of their diſtances at any two given places is inſenſible.

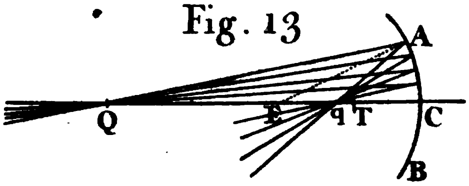
Reflection of a pencil of parallel rays at a plane ſurface.

22. The 7th and 8th figures repreſent a pencil of rays, *QC*, which falling in parallel lines upon a plane poliſhed ſurface, repreſented by the line *ACB*, are reflected from it into as many other parallel lines,

Cq;

[a] Art. 6.

Fig. 13



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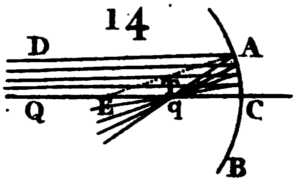

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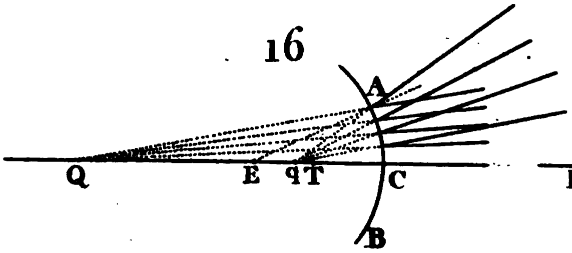

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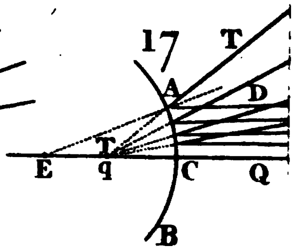

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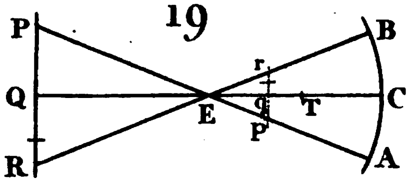

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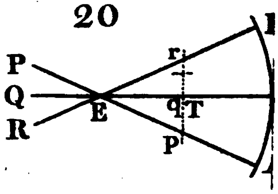

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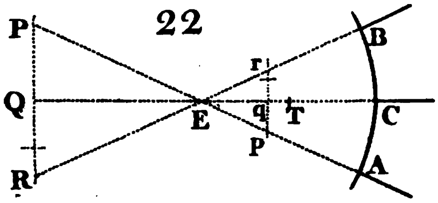

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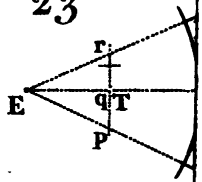

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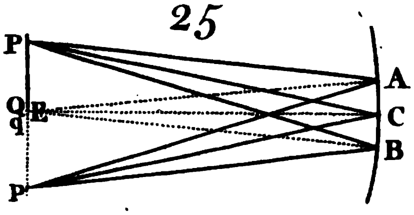

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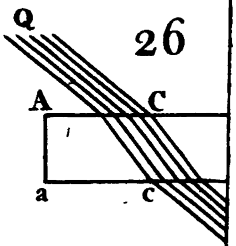

Cq; which are inclined to that plane juſt as much as the incident rays were inclined to it[a].

And of diverging rays.

23. The 9th figure repreſents the manner in which the rays of a pencil *QAB*, diverging from any point of an object *Q*, and falling upon a ſtraight line *ACB*, or upon a poliſhed plane repreſented by it, do all diverge after reflection as if they came from another point *q*. The ray *QC*, which falls perpendicularly upon the plane *AB*, is reflected back again along the ſame[b] line *CQ*; but all the reſt falling upon it with greater and greater degrees of obliquity as the points of incidence lye farther and farther from *C*, are alſo reflected with degrees of obliquity reſpectively greater[c]. It will ſeem reaſonable therefore, eſpecially by attending to the figure, that the reflected rays, produced backwards, ſhould meet the perpendicular *QC*, produced, in a point *q*, ſituated as far from the reflecting plane on one ſide, as *Q* is on the other: and conſequently that all the rays flowing from a ſingle point *Q*, will after reflection diverge from a ſingle point *q* at an equal diſtance on the other ſide of the reflecting plane.

And of converging rays.

24. On the contrary if *q* be a focus to which the incident rays are made to converge, by ways hereafter deſcribed, the point *Q* will be their focus after reflection[d] from the ſurface *ACB*.

And of ſeveral pencils which form images. Fig. 10, 11, 12.

25. What has been ſaid of the point *Q* is applicable to every other point of an object *PQR*: namely that as the focuſes *Q*, *q* lye at equal diſtances on each ſide of the reflecting plane, ſo the focuſes *P*, *p* lye on each ſide at other equal diſtances, and *R*, *r* at other equal diſtances, in lines *Pp*, *Rr*, drawn perpendicularly through the plane *AB*. Hence it is eaſy to underſtand by inſpection of the figures, that theſe focuſes *p*, *q*, *r*, with innumerable others, lying all in the ſame order as the correſponding points *P*, *Q*, *R*, compoſe an imaginary line of the ſame length and ſhape as the line *PQR*; and that the ſituation of the line *pqr*, with reſpect to the backſide of the reflecting plane, is the very ſame as that of *PQR* with reſpect to the foreſide of it. This line *pqr* is called an image or picture of the object *PQR*.

Reflection of a pencil of parallel rays at a ſpherical ſurface. Fig. 14, 17.

26. The 14th and 17th figures ſhew that parallel rays falling upon an arch of a reflecting circle *ACB*, or upon a concave or convex ſurface repreſented by it, are ſo reflected as to converge to a focus *T*, when they fall upon the concave ſide of the ſurface, and to diverge from *T* when they fall on the convex ſide. In both caſes the ray *QC*, which paſſes through, *E*, the center of the ſurface, and falls perpendicularly upon it at *C*, is reflected back again along the ſame[e] line *CQ*; but all other rays being parallel to *QC* fall upon the ſurface with various degrees of obliquity, by reaſon of its continued curvity. Every ray as it is more remote from *QC*, makes a greater angle of incidence *DAE*, with the perpendicular *EA* at the point of incidence; and conſequently

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[a] Art. 9.

[b] Art. 10.

[c] Art. 9.

[d] Art. 11.

[e] Art. 10, 19.

the equal [a] angle of reflection EAT becomes greater continually as the point of incidence, A, is farther from C. It is reaſonable therefore that all the reflected rays ſhould converge and convene pretty cloſe together about ſome certain point T of the direct ray QC, if the reflecting ſurface be concave, or elſe diverge from it, if the ſurface be convex. By reaſoning farther, as well as by experience, it is found that in both caſes the focus T divides the ſemidiameter CE into two equal parts.

[a] Art. 8, 19.

Reflection of diverging and converging rays. Fig. 13. to 18.

27. In the foregoing caſes, if the ſaid point T be a focus of incident rays, the reflected rays will all be parallel to the line CTE drawn through the center E [b]. But if the focus T be removed to any point q towards E, the angles of incidence, as qAE, and conſequently the equal angles of reflection, as EAQ, will all be diminiſhed; and if q be removed towards C, they will all be increaſed. Conſequently the reflected rays, as AQ, which before were parallel to the direct ray EC, will now be inclined to it; ſo as to belong to another focus Q ſituated on the ſame ſide of T as q. From the contemporary decreaſe of the angles of incidence and reflection while q is moved from T towards E, and their contemporary increaſe while it is moved towards C, it follows that the focuſes q, Q move contrary ways, ſo as to meet each other at the center E, or at the ſurface C, if the arch AC be very ſmall. It is obſervable that the properties of concave and convex ſurfaces are juſt alike, and are changed into one another by conceiving the incident rays to come the contrary way in the ſame lines produced.

[b] Art. 11.

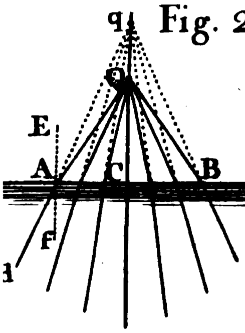
Reflection of ſeveral pencils which form images. Fig. 19. to 24.

28. Theſe figures ſhew the manner in which the image or picture pqr, of an object PQR, is formed by rays reflected from a concave or convex ſurface ACB. As the focus q has been ſhewn to lye in the perpendicular ray QC, drawn from Q through the center E, juſt ſo the focus p, of the pencil of rays that flow from another point P, will be in the perpendicular ray PA, drawn through the center E. For every ray that paſſes through the center or tends towards it, falls perpendicularly upon the arch or ſurface ACB, and all others fall obliquely upon it.

Some general properties of images.

29. Hence it is eaſy to underſtand that if the object PQR be ſo ſmall, or ſo far diſtant from the reflecting ſurface or its center E, that all the points P, Q, R be nearly at equal diſtances from it, then all the points p, q, r, of the image, will be nearly at other equal diſtances from the ſaid ſurface or its center. It is alſo to be obſerved, that when the object and its image lye both on the ſame ſide of the center, the image is upright, but when they lye on contrary ſides of the center, it is inverted; and that it is greater or leſs than the object in proportion as it lies at a greater or leſs diſtance from the center than the object it ſelf. Theſe things are plain by inſpection of the figures, obſerving that the object and image are both terminated by two lines Pp, Rr, that meet or croſs in the center E. Hence it follows that the image will be nearly equal

Fig. 28



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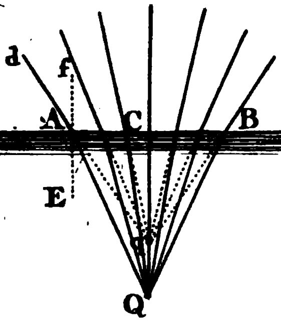

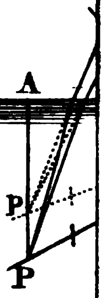

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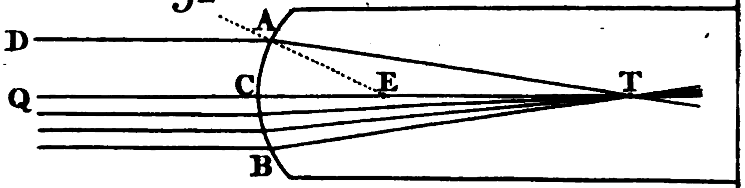

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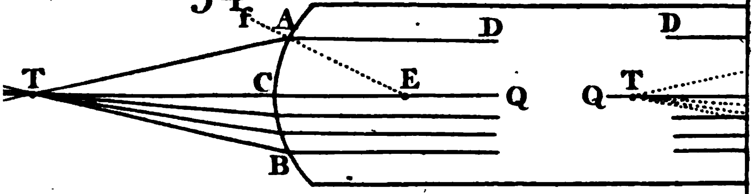

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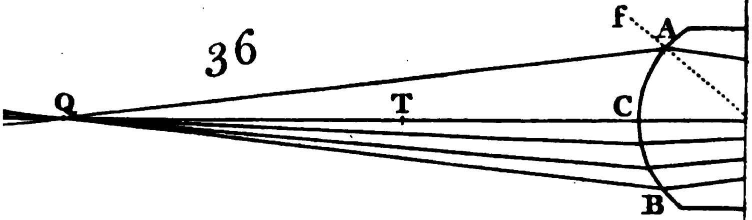

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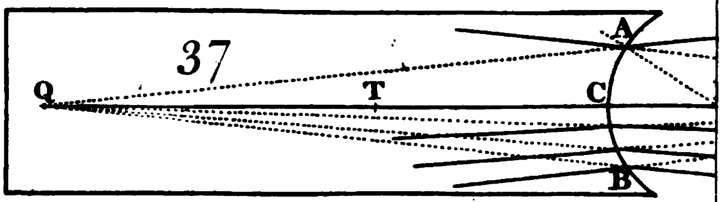

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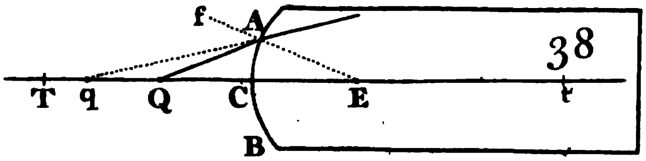
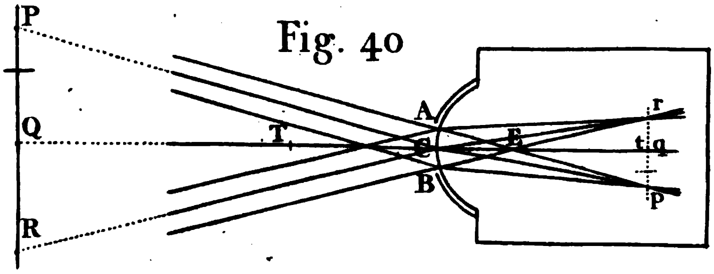


Fig. 40



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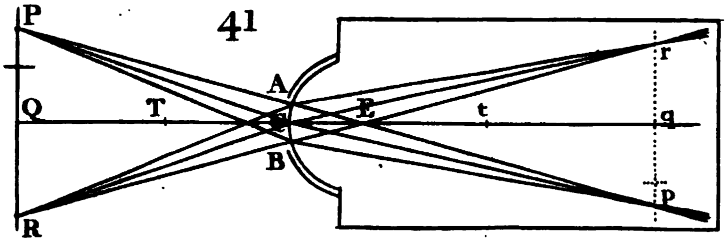

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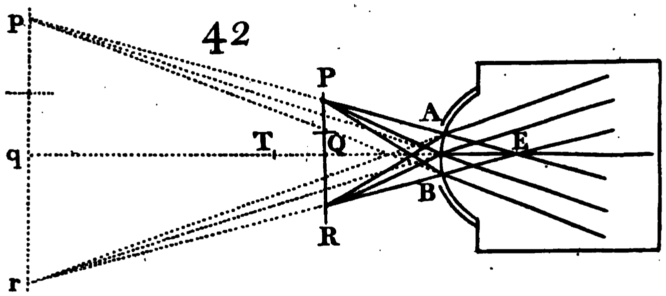

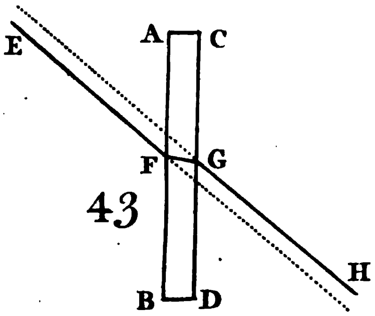
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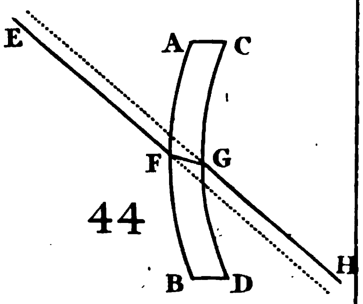
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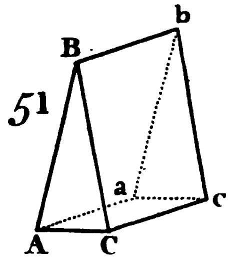
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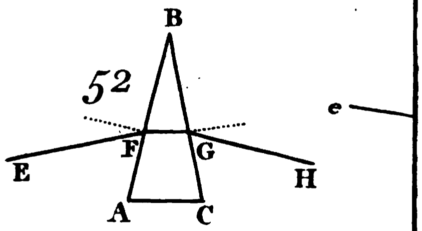
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equal to the object when they meet[a] each other at the ſurface, and alſo when they meet at the center. For in this latter caſe the rays which flow from Q placed in the center, will be reflected directly back to q in the ſame center; and taking Ep equal to EP, ſince EC is perpendicular to them both, the angles PCE, ECp will be equal; and ſo the ray PC will be reflected to p: and when any other point of incidence as A is not far from C, the line AE will alſo be nearly perpendicular to Pp, and ſo the angles PAE, EAp being nearly equal, the ray PA will be reflected nearly to the ſame point p as the ray PC was reflected to. I proceed now to refractions.

Fig. 25.

Refraction of a pencil of parallel rays at a plane ſurface.

30. In the 26th figure QC repreſents a pencil of parallel rays falling obliquely upon a ſtraight line ACB, or upon a plane ſurface repreſented by it, which after refraction are alſo parallel among themſelves; every one being equally bent. Becauſe when the angles of incidence are all equal among themſelves the angles of refraction are alſo equal among themſelves[b]. For the ſame reaſon if theſe rays be refracted again at another plane, either parallel or oblique to the former, they will ſtill be parallel among themſelves after every refraction. In ſtrictneſs this is only to be underſtood of rays of the ſame colour: as will be explained in the 6th chapter.

Fig. 26, 27.

Refraction of a pencil of diverging and converging rays.

31. The rays of a pencil QAB, diverging from Q, and falling upon a ſtraight line ACB, or upon a plane ſurface repreſented by it, are ſo refracted as to diverge from another point q ſituated in that ray QC (produced) which falls perpendicularly upon the plane. For this ray paſſes ſtraight through the ſurface[c], but all the reſt, as QA, are bent; and every one ſo much the more as the point of incidence A is remoter from C[d]; becauſe the angle of incidence QAE, and conſequently that of refraction grows larger[e]. For this reaſon all the refracted rays will diverge pretty nearly from a certain point q on the ſame ſide[f] of the ſurface AB as Q. It is found by other arguments and alſo by experience, that if the refracting body be glaſs, the greater of the two focal diſtances QC, qC, is to the leſſer as 3 to 2; and if it be water, as 4 to 3; that is, in the proportion of the ſines of incidence and refraction in thoſe ſeveral mediums[g]. On the contrary if the incident rays be made to converge towards q, the refracted rays will converge to Q[h].

Fig. 28, 29.

Refraction of ſeveral pencils which form images.

32. The 30th and 31ſt figures repreſent an image pqr of an object PQR, formed by a refracting plane ACB, in the manner deſcribed in the 25th article. The ratios of Ap to AP, Br to BR, &c. are all equal.

Refraction of a pencil of parallel rays at a ſpherical ſurface.

33. The 32d, 33d, 34th and 35th figures ſhew in what manner a pencil of parallel rays falling upon an arch of a circle ACB, or upon a ſpherical ſurface repreſented by it, do after refraction converge to or diverge from a focus T. The ray QC which paſſes through E, the center of the ſurface, and conſequently falls perpendicularly upon it,

[a] Art. 27.

[b] Art. 14.

[c] Art. 15.

[d] Art. 16.

[e] Art. 14.

[f] Art. 11, 12.

[g] Art. 13.

[h] Art. 11.

goes ſtraight through it without refraction[a]. But all the reſt, being parallel to QC, fall with various degrees of obliquity upon the ſurface by reaſon of its continued curvity; and every one of them as it is remoter from QC falls more and more obliquely and conſequently is more and more bent[b]. It is reaſonable therefore that the refracted rays ſhould converge and convene pretty cloſe together about ſome certain point T, in the unrefracted ray QC (produced), if they bend towards that ray; or elſe diverge from T, if they bend from it. Which way they are bent will appear by drawing a perpendicular EA to the ſurface at A, and by conſidering the poſition of the denſer medium[c]. Hence it will appear that if the ſurface of the denſer medium be convex, the refracted rays will converge to T; and diverge from T, if it be concave. It is found by farther reaſoning, and alſo by experience, that if the refracting body be glaſs, the greater of the two diſtances CT, TE is to the leſſer as 3 to 2; and as 4 to 3 if it be water or ice; that is, in the proportion of the ſines that determine the refractions in thoſe bodies.

Refraction of a pencil of diverging and converging rays at a ſpherical ſurface.
Fig. 36 to 39.

34. In the foregoing caſes if T be a focus of incident rays the refracted rays will be parallel to the perpendicular ray TC, that paſſes through E[d]. But if this focus T be removed to any other point Q in the line TC produced, the angles of incidence and refraction will either be increaſed all together, or diminiſhed all together; and conſequently all the refracted rays, which before were parallel to TC, will now be inclined to it; ſo as to belong to another focus q, on the contrary ſide of the ſurface to Q, if Q be farther from the ſurface than T; otherwiſe on the ſame ſide of the ſurface. One of the chief properties of theſe correſponding focuſes Q, q, is this that follows. Since the angles of incidence and refraction do both increaſe or elſe both decreaſe together[e], it follows that the focuſes Q, q muſt both move the ſame way in the line Fig. 38, 39. QE produced. And conſequently, while both lye on the ſame ſide of the ſurface or of its center, they muſt both move from it or both towards it; and if they move towards it, they will come nearer together; till they both coincide at the center or both at the ſurface, when the Fig. 36, 37. arch AC is very ſmall. But if the focuſes Q, q be on contrary ſides of the ſurface or of its center; while one moves from it, the other will move towards it; and on the contrary.

Refraction of ſeveral pencils which form images at a ſpherical ſurface.

35. The 40th, 41ſt and 42d figures ſhew in what manner the image pqr of an object PQR is formed by ſeveral pencils of rays, (refracted by a ſpherical ſurface,) whoſe axes or unrefracted rays are PEp, QEq, REr. The properties of theſe images are the ſame as of thoſe made by reflection from a ſpherical ſurface, and are already deſcribed in the 29th article.

Refraction of a ray through parallel plane ſurfaces.

36. A ray of light EF falling obliquely upon a flat piece of glaſs, or any medium terminated by two parallel planes repreſented by the

lines

[a] Art. 15.

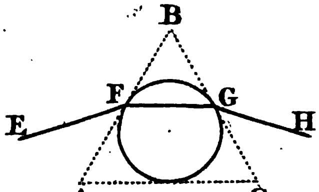
[b] Art. 16.

[c] Art. 12.

[d] Art. 11.

[e] Art. 14.

Pag. 11.



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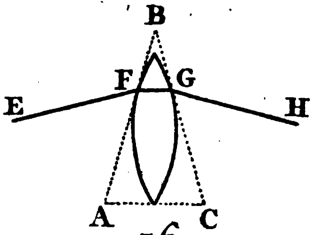

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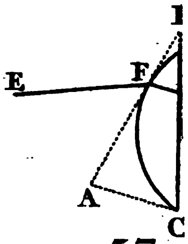

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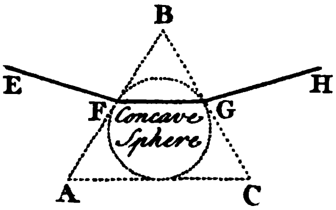

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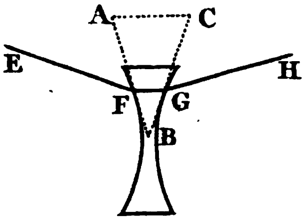

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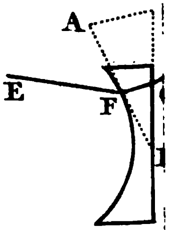

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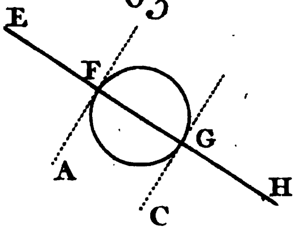

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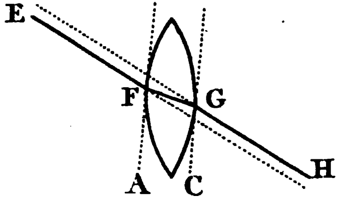

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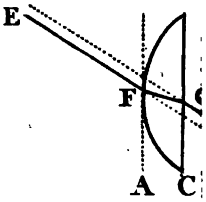

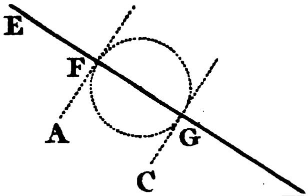

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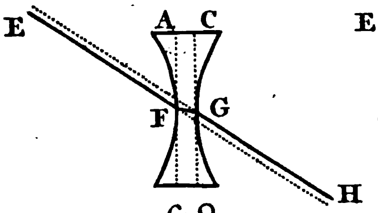

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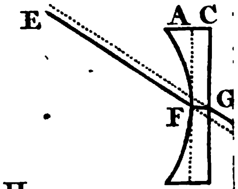

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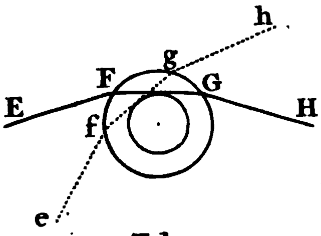

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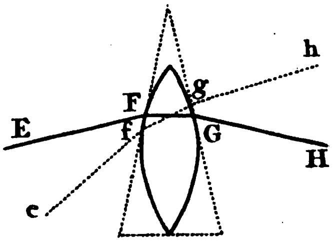

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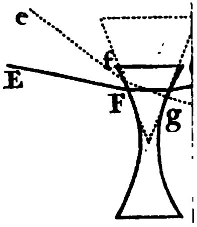

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lines *AB*, *CD*, will emerge from it after both refractions at *F* and *G* in a line *GH* parallel to the incident ray *EF*. For ſince any line *FG* which the ray deſcribes in paſſing between the parallel planes, is equally inclined to them both, it will be bent as much at *G* in going forward, as it would be at *F* in going backward[a]; and theſe equal bendings being made contrary ways, the incident and emergent rays *EF* and *GH* are therefore parallel.

Fig. 43.

[a] Art. 11.

Refraction of a ray through parallel ſpherical ſurfaces.

37. The lines deſcribed by the incident and emergent rays *EF* and *GH*, being produced are cloſer together when the glaſs is thinner, and alſo when the ray falls leſs obliquely upon it; becauſe the bendings at *F* and *G* are then leſs[b]: and in theſe caſes if the glaſs be not flat but bent a little as repreſented in the 44th figure by two parallel arches *AB*, *CD*, the lines *EF*, *GH* will ſtill be nearly parallel. For the bended ſurfaces refract the ray *EFGH* juſt as much as two planes would do ſuppoſing they touched the ſurfaces at *F* and *G*[c]: and theſe planes will be nearly parallel when the line *FG* is but little inclined to the ſurfaces; being exactly ſo when it ſtands perpendicular to them both.

[b] Art. 16.

[c] Art. 19.

A lens what.
Fig. 45 to 50.

38. A thin piece of glaſs or of any tranſparent ſubſtance bounded on one ſide by a poliſhed plane ſurface, repreſented by the line *EF*, and on the other ſide by a ſmall portion of a poliſhed ſpherical ſurface, repreſented by the arch *ACB*; or bounded on both ſides by ſpherical ſurfaces *ACB*, *EDF*, is called a lens or ſimply a glaſs; and is conceived by mathematicians to be generated or deſcribed by turning the figure *ACBFDE* round about the line *CD*, drawn through the middle of it perpendicularly to both its ſides. This line *CD* produced is therefore called the axis of the lens; and it paſſes through *G* and *H*, the centers of its ſurfaces. The points *C*, *D* where it cuts the ſurfaces are called the vertexes of the lens, and the middle point between them is called its center. The 45th figure repreſents a plano-convex glaſs, the 46th a plano-concave, the 47th a double-convex, the 48th a double-concave, and the 49th and 50th two concavo-convex glaſſes, whereof the firſt is called a meniſcus, becauſe it reſembles a little moon. It muſt be remembered once for all, that the thickneſs *CD* of all theſe glaſſes is generally ſo ſmall, that it ſeldom need be conſidered.

A priſm what.
Fig. 51.

39. A glaſs priſm is a body, ſhaped like a wedge, that has three edges, being bounded with two equal and parallel triangular ends *ABC* and *abc*, and three plane and well poliſhed ſides, which meet in three parallel lines *Aa*, *Bb*, *Cc*, running from the three angles of one end to the three angles of the other: and when it is viewed endways it is repreſented only by a triangle *ABC*, as in the 52d figure.

Refractions of a ſingle ray through a priſm.
Fig. 52, 53, 54.

40. When a ray of light *EFGH* is refracted at *F* and *G* in paſſing through the ſides, *AB*, *BC*, of a priſm, the courſe of the emergent ray, *GH*, always deviates from, *EF*, the courſe of the incident ray, to-

wards the thicker part of the priſm, more or leſs, as the refracting angle *ABC* is greater or ſmaller. And if the refracting angle be given (or invariable) and the refractions be but ſmall, the quantity of deviation will alſo be given, though the poſition of the incident ray be varied at pleaſure.

Fig. 52. For ſuppoſing at firſt that the ray *FG*, within the priſm, is equally inclined to its ſides *AB*, *BC*, as in fig. 52, it is evident from the poſition of the perpendiculars to thoſe ſides at the points *F* and *G*, that both the refractions are made from the edge *B* towards the oppoſite ſide *AC*[a].

Fig. 53. Now let *FG* become unequally inclined to the ſides *AB*, *BC*, by turning it gradually into the poſition *fg*; and while it becomes leſs and leſs oblique to one ſide, ſuppoſe *AB*, it will become more and more oblique to the other ſide *BC*. Conſequently ſuppoſing a ray to go both ways along this variable line *fg*, it will be more and more bent in going through the ſide *BC* and leſs and leſs in going back through the ſide *AB*; ſo that the total bending of the ray, conſiſting of both its bendings, or angles *efg* and *fgh*, taken together, will continue to be much the ſame in all its poſitions. The circulation of the line *fg*, may be farther continued till it becomes perpendicular to the ſide *AB*; and Fig. 54. then the bending at this ſide is nothing: it may alſo be continued ſtill farther till the bending at *f* is made the contrary way; which ſtill takes off from the perpetual increaſe of the greater bending at *g* and keeps the total bending invariable.

Fig. 53. When *fg* is perpendicular to *AB*, let the latter plane *BC* be turned gradually towards the former *BA*, upon the edge *B*, and the ray that comes along *fg* will gradually fall leſs obliquely upon it; and conſequently the bending at *g* will be gradually diminiſhed[b]; and reduced to nothing when the refracting angle *ABC* vaniſhes. Laſtly if ſeveral rays be ſuppoſed to come parallel to one another they will all emerge parallel to one another[c]. Therefore the quantity of deviation of a ray does not at all depend upon its paſſage through a thicker or thinner part of the priſm, nor upon its inclinations to the ſides of the priſm, but is proportioned to the quantity of the refracting angle *ABC*; and the more exactly as this angle and the refractions at its ſides are ſmaller.

Refractions of a ſingle ray through the edge of a lens. Fig. 55 to 62.

41. For the ſame reaſon when a ray of light *EFGH* paſſes through the edge of a convex or concave lens, or the ſides of a globe, its emergent part *GH* always deviates from the courſe of the incident part *EF* towards the thicker part of the glaſs. Becauſe the refractions at *F* and *G* are the ſame as if they were made by two planes *FA*, *GC* that touch the ſpherical ſurface at *F* and *G*[d]; and ſo the ſides of the glaſs may be conſidered as inclined to each other like the ſides of a priſm.

Refractions of a ſingle ray through the middle of a lens.

42. It follows therefore from the two laſt articles, that the deviation of the courſe of the emergent ray, from that of the incident ray, is gradually

[a] Art. 12.

[b] Art. 14, 15.

[c] Art. 30.

[d] Art. 19.

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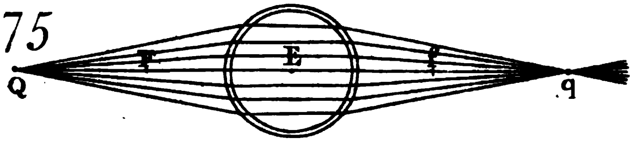

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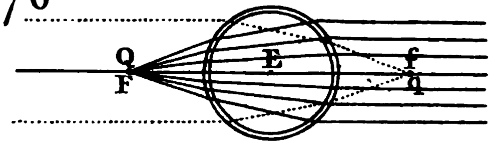

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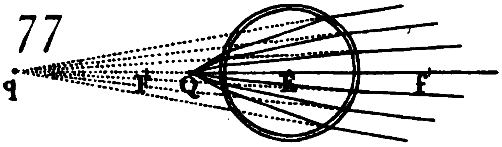

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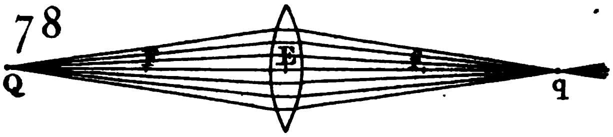

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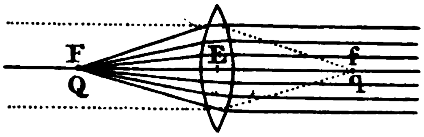

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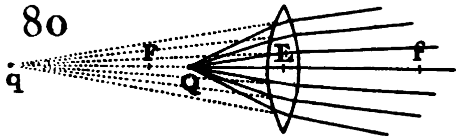

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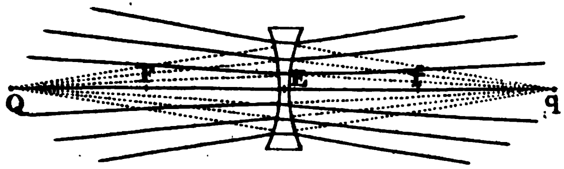

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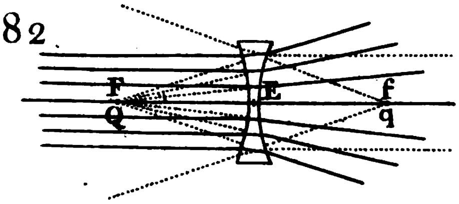

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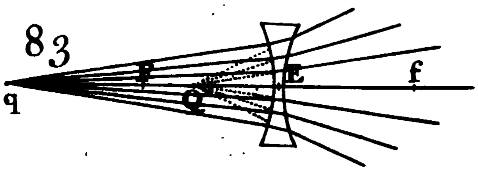

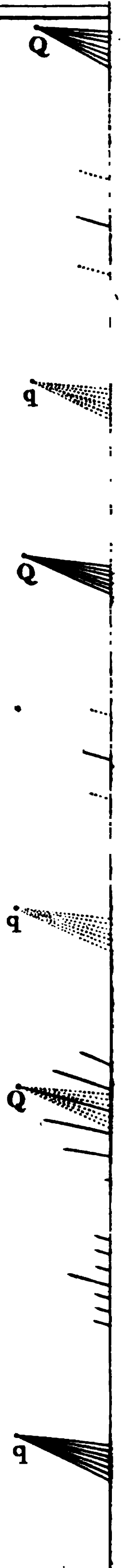

gradually diminiſhed as the ray goes nearer and nearer to the middle of the glaſs: till, when it goes through the middle, its emergent and incident parts are either parallel to each other, or elſe are one continued line, when the ray coincides with the axis of the glaſs. For the angle made by the touching planes, *FA*, *GC*, is gradually diminiſhed as the ray *FG* approaches to the middle: till at laſt it vaniſhes when they become parallel, as in the 36th article.

Fig. 63 to 70.

43. When a pencil of rays falls upon any glaſs, that ray which paſſes through its center, or middle point, is called the axis of the pencil. And becauſe its incident and emergent parts, *EF* and *GH*, are either one continued line or two parallel lines [a], its whole courſe in optical experiments may be always taken for one ſtraight, phyſical line: from which it differs inſenſibly when the thickneſs of the glaſs is ſmall, and when the pencil falls not too obliquely upon it. Becauſe the parallel lines *EF* and *GH* produced, go cloſer together in proportion as the line *FG* is ſhorter, and as the bendings at *F* and *G* are ſmaller.

This ray is conſidered as ſtraight, and is called the axis of a pencil.

44. All rays, as *EFGH* and *efgh*, which croſs each other in a refracting globe and paſs through it at equal diſtances from its center, ſo as to touch a concentrick globe, are equally bent. For in this caſe the chords *FG*, *fg* being equal, their obliquities to the ſurface of the globe are alſo equal, and conſequently the bendings of the ray *EFGH* at *F* and *G*, both ſeverally and together, are equal to the bendings of the ray *efgh* at *f* and *g*: as is evident by conceiving the rays to go both ways along the chords *FG*, *fg*. Therefore the angle made by the incident and emergent parts of one ray, produced till they meet, will be equal to the angle made by the incident and emergent parts of the other ray produced till they meet; which is what I mean when I ſay the rays are equally bent.

Rays are equally bent which paſs at equal diſtances from the center of a globe. Fig. 71, 74.

45. All rays, as *EFGH*, *efgh*, which croſs each other at any given point of a lens, or which paſs through it at equal diſtances from its center, are equally bent, provided they do not fall very obliquely upon it. Imagine a line *FG* within the glaſs, at firſt to be equally inclined to its ſides, and then to be turned a little about any point of it, till it comes into the poſition *fg*; and while it becomes more and more oblique to one ſide of the glaſs, ſuppoſe *Ff*, it will become leſs and leſs oblique to the other ſide *Gg*. Conſequently if a ray be ſuppoſed to go both ways along this variable line *fg* it will be more and more bent in going through the ſide *Ff* and leſs and leſs bent in going through the other ſide *Gg* [b]: ſo that the total bending of the ray conſiſting of both its bendings, or angles *efg*, *fgh*, taken together, will continue to be much the ſame in all its poſitions. The circulation of the line *fg*, about the given point, may be farther continued till the bending at *g* is diminiſhed to nothing; and ſtill farther till it be made the contrary

And from the center of a lens. Fig. 72, 73.

way;

[a] Art. 42.

[b] Art. 16.

way; (as was explained in the 40th article;) which ſtill takes off from the perpetual increaſe of the greater bending at *f* and keeps the total invariable. To keep the ſame bending it is only neceſſary that the rays *FG*, *fg* ſhould keep at equal diſtance from the axis of the lens as near as poſſible: and nothing alters the total bending but the alteration of that diſtance[a]; becauſe the inclination of the tangent planes, like the refracting angle of a priſm, will then only be altered.

[a] Art. 40.

Refraction of a ſingle pencil of parallel rays by any glaſs. Fig. 76, 79, 82, 85, 88, 91.

46. When a large pencil of parallel rays falls either directly or a little obliquely upon the whole ſurface of any glaſs, which is thicker in the middle than at the edges; all the emergent rays will be bent from all ſides towards that ray which goes through the middle of the glaſs: and on the contrary, if the glaſs be thicker at the edges than at the middle, they will all be bent outwards from the middle ray[b]. And becauſe in both caſes the bendings are equal at all equal diſtances round about the middle; and grow greater at greater diſtances from the middle[c]; the emergent rays will all converge pretty nearly to ſome certain point *F* in the middle ray, if the glaſs be convex; or diverge from a certain point *F*, if the glaſs be concave.

[b] Art. 41.

[c] Art. 40.

Refraction of parallel rays coming contrary ways.

47. When parallel rays come contrary ways and fall upon oppoſite ſides of any lens, the diſtances of the focuſes of the emergent rays from each ſide of the center of the lens will be equal; though the ſemidiameters of the ſurfaces of the lens be never ſo unequal, or though one ſide be plane and the other ſpherical. For ſince any two rays which come directly oppoſite to each other, are at equal diſtances from the common axis of the pencils, and after croſſing each other and emerging from the glaſs, are equally bent from their firſt courſe[d], theſe two emergent rays (produced) will meet the axis at equal diſtances *EF*, *Ef* from the center of the glaſs. When the rays come parallel to the axis of the lens, their two focuſes *F*, *f* are called the principal focuſes of the lens: and *EF* or *Ef* is called its focal diſtance, and by ſome authors its focal length.

[d] Art. 44, 45.

Principal focus, and focal diſtance, what.

Refraction of a ſingle pencil of diverging or converging rays through any glaſs. Fig. 75 to 92.

48. On the contrary if the rays be returned directly back from the focus *F*, the emergent rays will all be parallel to *FE*, the axis of the pencil[e]. Conſequently if this focus *F* be removed to *Q*, farther from the glaſs, the emergent rays will belong to another focus *q* on the contrary ſide of the glaſs; but if *Q* be put nearer to the glaſs than *F*, the emergent rays will belong to a focus *q* on the ſame ſide of the glaſs as *Q*. Becauſe while the rays are put into theſe different ſituations their bendings will not be altered, if they keep their reſpective diſtances from the center of the glaſs[f]. Conſequently if either of the correſponding focuſes *Q*, *q* be put in motion along the axis of the pencil, either direct or oblique, the other focus will move the ſame way: and therefore if theſe focuſes be on contrary ſides of the glaſs, while one moves towards

[e] Art. 11.

[f] Art. 44, 45.

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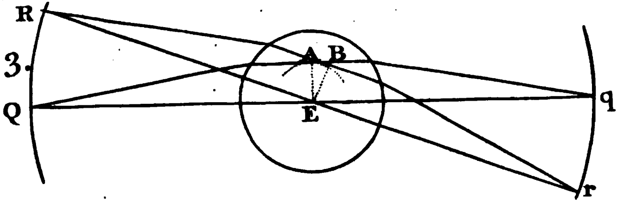

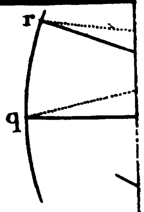

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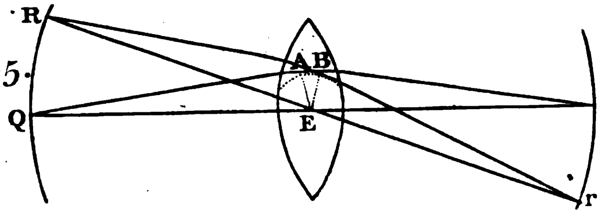

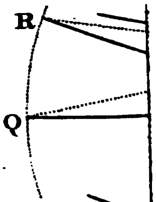

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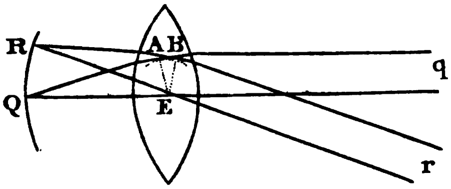

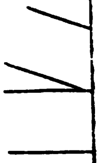

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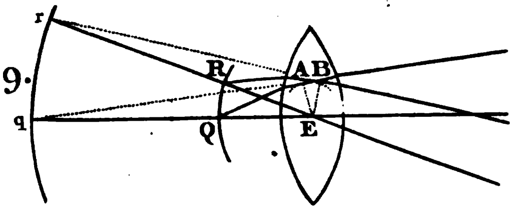

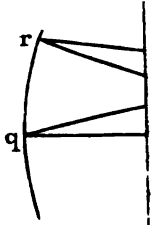

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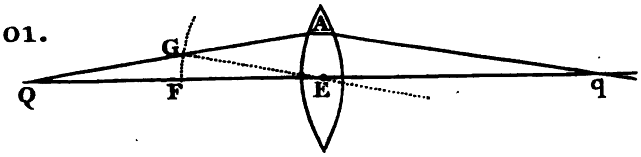

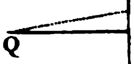

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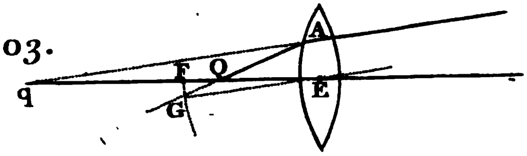

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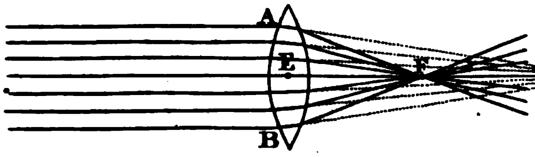

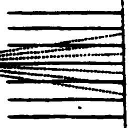

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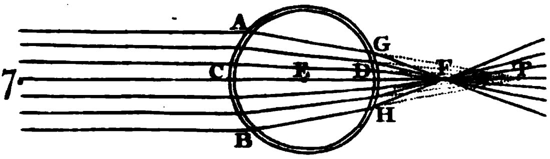

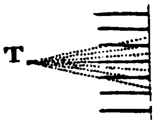

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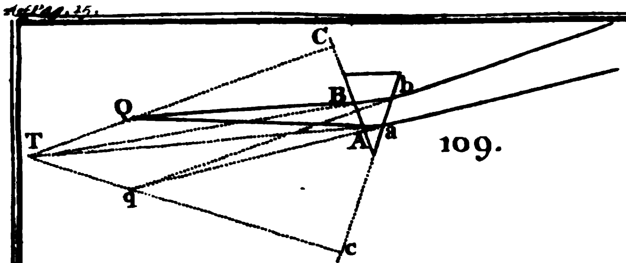

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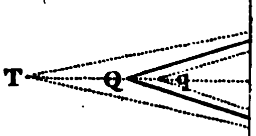

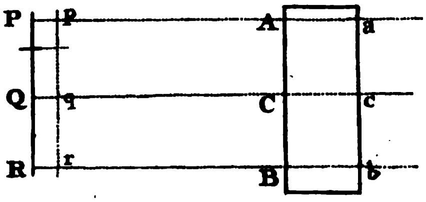

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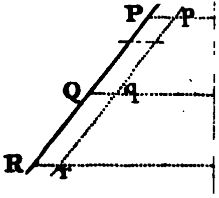

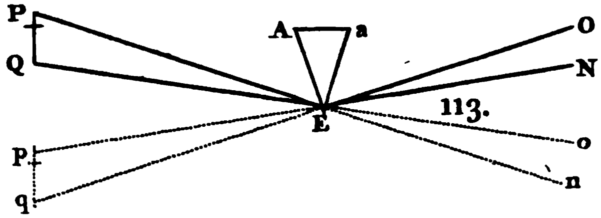

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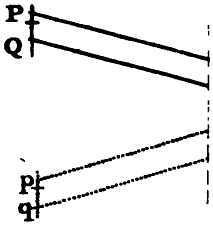

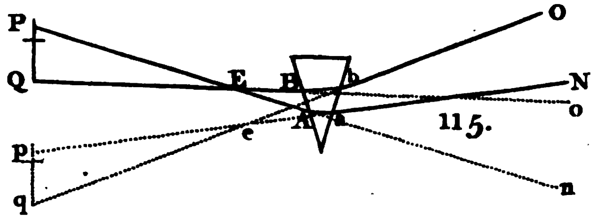

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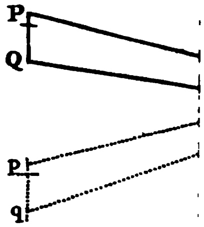

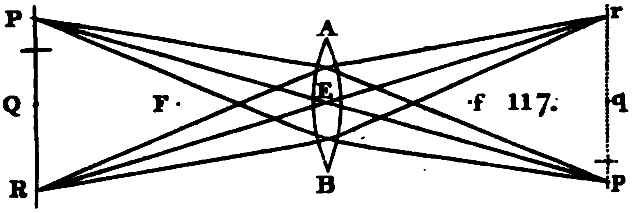

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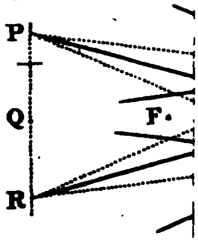

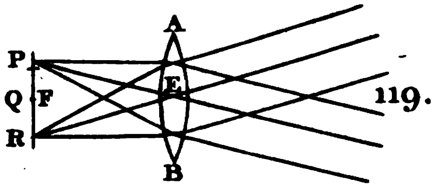

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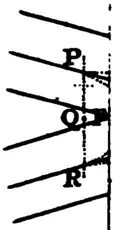

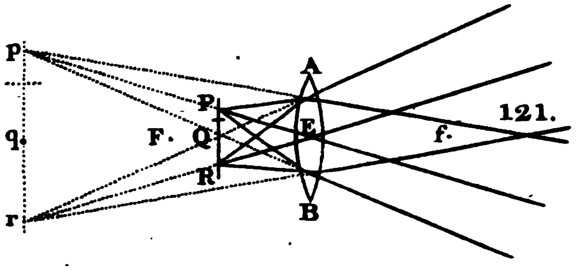

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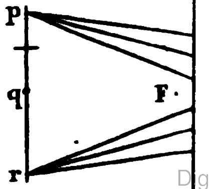

towards it the other will move from it; but if they be both on the ſame ſide of the glaſs, they will both move from it or both towards it; and will come nearer to each other as they come nearer to the glaſs, till when one coincides with its ſurface the other will do ſo too very nearly, provided the glaſs be very thin and the diſtance of the ray from its axis be very ſmall. Theſe focuſes cannot therefore coincide at the ſurface of a globe, becauſe the points of incidence and emergence are too far aſunder.

It is obſervable that the properties of concave ſurfaces and glaſſes are the ſame as of convex ones; which will appear by conceiving the rays to come contrary ways in the ſame lines produced; and accordingly to be changed from diverging to converging, and on the contrary: as repreſented in the figures, by the black lines and pointed lines.

Refractions of ſeveral pencils of diverging and converging rays. Fig. 93 to 100.

49. If ſeveral focuſes Q, R of incident rays be at any equal diſtances EQ, ER from the center of any glaſs, the focuſes of the emergent rays will alſo be at other equal diſtances Eq, Er from the ſame center in the lines EQ, ER produced; provided none of the rays fall very obliquely upon the glaſs. Take any point A within the glaſs, not far from its axis Qq, through which a ray paſſes from the focus Q to the focus q. Draw the line AE and while the figure $QAEq$ is conceived to turn a little about the center E, into the poſition $RBEr$, the extremities of the lines EQ, EA, Eq will deſcribe ſmall arches QR, AB, qr about that common center E. Then let another ray which belongs to the focus R be refracted through the point B, and after emergence it will belong to the point r; becauſe the total bendings of two rays, QAq, RBr, which paſs at equal diſtances, AE, BE, from the center of the glaſs, are equal [a]. And the reſt of the rays that belonged to R will belong to the ſame point r; becauſe it is ſituated in the axis of the pencil [b].

Refraction of ſeveral pencils of parallel rays. Fig. 97, 98.

50. Hence the focuſes of all pencils of parallel rays that fall not too obliquely on the ſame or on oppoſite ſides of any glaſs are equidiſtant from its center. For the argument continues the ſame while the equal diſtances Eq, Er, are equally increaſed till they become infinite, that is till the rays of each pencil become parallel.

Having the focus of incident rays upon a lens to find the focus of the emergent rays. Fig. 101 to 104.

51. Hence if Q, the focus of incident rays be given, and q, the focus of the emergent rays be required, draw QE, the axis of the pencil, and with the center E and ſemidiameter EF, equal to the focal diſtance of the lens, (to be found by experiment,) deſcribe an arch FG cutting any incident ray, QA, in G; join EG and drawing Aq parallel to it, the point q where it cuts the axis of the pencil will be the focus of the emergent rays. For ſuppoſing other rays, beſides GA, to flow from or towards G, they will all emerge parallel to their axis GE produced [c].

[a] Art. 44, 45.

[b] Art. 46.

[c] Art. 50.

52. The

Refraction of rays through ſeveral ſurfaces otherwiſe conſidered. Fig. 105 to 110.

52. The refraction of a pencil of rays through all ſorts of glaſſes may alſo be conſidered in this manner. By the refraction at the firſt ſurface AB the rays are diſpoſed within the glaſs to converge to or diverge from a focus T; which may be conſidered as the focus of incident rays upon the ſecond ſurface; by whoſe refraction they are all directed to another focus F. For example, let Q be the focus of incident rays upon a glaſs priſm; QC perpendicular to its firſt ſide AB. To QC add QT, equal to half QC; and T will be the focus of the rays QA, QB, &c. after refraction at the ſurface AB[a]; and being alſo the focus of incident rays at a and b upon the ſecond ſurface ab, from Tc drawn perpendicular to ab take away Tq equal to a third part of Tc; and q will be the focus of the emergent rays qa, qb produced.[a]

[a] Art. 31.

Hence the focuſes of incident and emergent rays at a priſm, lye always very nearly at equal diſtances from it; provided the refractions and the refracting angle be but ſmall. For then the perpendiculars TC, Tc are nearly equal; and in glaſs QC and qc are two thirds of them reſpectively.

Fig. 110.

Hence when the planes AB, ab are parallel, TC and Tc coincide; and Qq is one third of Cc, the thickneſs of the glaſs.

Images formed by a flat piece of glaſs. Fig. 111, 112.

53. An image pqr, formed by a flat piece of glaſs $AB\ ba$ is upright, parallel and equal to the object, PQR, and lyes on the ſame ſide of the glaſs as the object; but nearer to it by a third part of the thickneſs of the glaſs: becauſe we have ſhewn that the focuſes p, q, r, of the ſeveral pencils that flow from P, Q, R, lye ſo much nearer; in the lines PA, QC, RB, drawn from the ſeveral points of the object, perpendicular to the glaſs.

Images formed by a priſm. Fig. 113.

54. An image formed by a priſm is always upright, and equal to the object, and lyes on the ſame ſide of the priſm, and at the ſame diſtance from it as the object it ſelf: provided the refracting angle of the priſm, and the refractions made by it, be but ſmall. Take two rays, PE, QE, which coming from the extremities of the object, paſs through a point E, ſo near to the angular point of the refracting angle, that the diſtances between their points of incidence and emergence need not be mentioned. And ſince the total bendings of the rays PEN, QEO are equal[b], they will croſs each other, ſo as that the angle PEQ will be equal to the angle NEO or to pEq, made by the emergent rays produced backwards: and becauſe the diſtance Ep of the focus p, of the pencil that flowed from P, is equal to EP[c], and in like manner the focal diſtance Eq equal to EQ; the image pq will be upright and equal to the object and at an equal diſtance on the ſame ſide of the priſm.

[b] Art. 40.

[c] Art. 25.

Fig, 114, 115, 116.

The ſame things might have been proved, by conceiving two rays PA, QB to go from the extremities of the object either parallel to one

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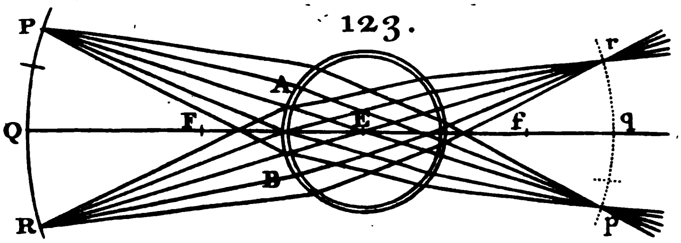

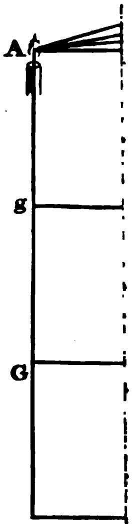

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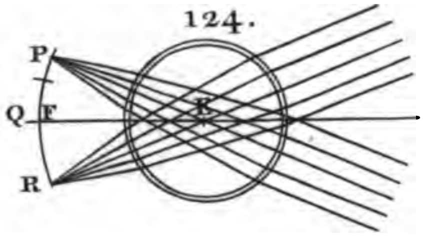

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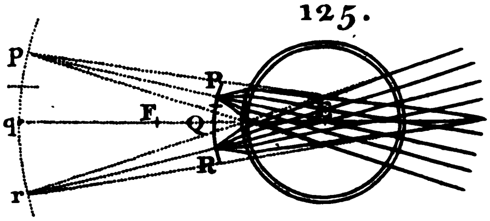

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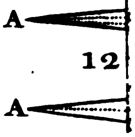

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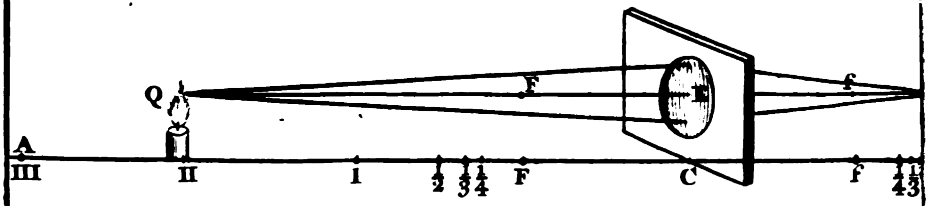

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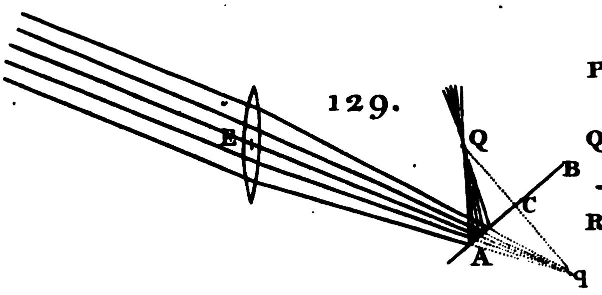

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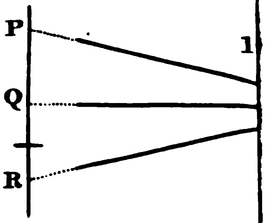


one another or ſo as to croſs at any point E. Becauſe the emergent rays (produced) will accordingly be parallel[a], or will croſs at another point e on the ſame ſide of the priſm, and at the ſame diſtance from it as E[b]; and will alſo make equal angles at E and e, as they did when they croſſed at the angular point of the priſm[c].

Images formed by glaſſes of all ſorts. Fig. 117 to 125.

55. Theſe figures repreſent the manner in which an image of an object is formed by ſeveral pencils refracted through a glaſs of any ſort. And becauſe the axes PEp, QEq, REr of the ſeveral pencils go, in a manner, ſtraight through the center of the glaſs, the properties of theſe images are the ſame as of thoſe which are made by refraction or by reflection at a ſingle ſurface, and deſcribed in the 29th article. Except that the image of an object which touches a globe does not coincide with the object it ſelf, but is ſituated at a diſtance from it, for the reaſon given at the end of the 48th article. According to theory the image of a circular arch will be nearly circular[d]; but when the object and image are ſmall and are placed at a conſiderable diſtance from the glaſs, the difference in their figures will be inſenſible in phyſical matters whether they be conſidered as circular arches or as ſtraight lines. Eſpecially conſidering that all the rays of a pencil do not croſs preciſely in a ſingle point of its axis, but in ſeveral points which make up a ſenſible part of it; as will appear by the following experiments.

Reflections and refractions at rough unpoliſhed ſurfaces.

56. When rays of light fall upon any rough unpoliſhed ſurface of an opake or tranſparent body, they are not reflected or refracted regularly, according to the laws and properties of poliſhed ſurfaces, but are ſcattered every way alike from the inequalities of the rough ſurface in the ſame manner as if they were emitted from a ſelf-ſhining ſubſtance.

A deſcription of ſome eaſy experiments, by which the truth of the foregoing properties of glaſſes may be readily examined, and ſome others be diſcovered.

I. Experiment. To ſhew that the breadths of a pencil are as their diſtances from the focus. Fig. 126.

57. Let the light which flows from a point A and paſſes through a ſquare hole $bcde$ be received upon a plane, $BCDE$, parallel to the plane of the hole; or if you pleaſe let the figure BD be the ſhadow of the plane bd; and when the diſtance AB is double of Ab, the length and breadth of the ſhadow BD will each be double the length and breadth of the plane bd; and treble, when AB is treble of Ab; and ſo on: which may be eaſily examined by the light of a candle placed at A.

Hence the denſity and quantity of light received upon a given plane are reciprocally as the ſquares of

58. Therefore the ſurface of the ſhadow BD, at the diſtance AB double of Ab, is diviſible into four ſquares, and at a treble diſtance, into nine ſquares, ſeverally equal to the ſquare bd, as repreſented in the figure. The light then which falls upon the plane bd, being ſuffered to paſs to a double diſtance, will be uniformly ſpread over four times

[a] Art. 50.

[b] Art. 52.

[c] Art. 40.

[d] Art. 49.

its diſtances from the luminous body.

the ſpace, and conſequently will be four times thinner in every part of that ſpace, and at a treble diſtance it will be nine times thinner, and at a quadruple diſtance ſixteen times thinner, than it was at firſt; and ſo on according to the increaſe of the ſquare ſurfaces *bcde*, *BCDE* &c, or of the ſquare ſurfaces *Abfg*, *ABFG*, &c, built upon the diſtances *Ab*, *AB*, &c. Conſequently the quantities of this rarified light received upon a ſurface of any given ſize and ſhape whatever, removed ſucceſſively to thoſe ſeveral diſtances, will be but one quarter, one ninth, one ſixteenth, of the whole quantity received by it at the firſt diſtance *Ab*. Or in general words the denſities and quantities of light, received upon any given plane, are diminiſhed in the ſame proportion as the ſquares of the diſtances of that plane, from the luminous body, are increaſed: and on the contrary, are increaſed in the ſame proportion as thoſe ſquares are diminiſhed. For the lights of the ſeveral points of the body, which ſeverally follow this rule, will compoſe a light which will ſtill follow the ſame rule.

Equal parts of a ſmall object ſubtend equal angles at the eye. Fig. 127.

59. When the perpendicular ſubtenſe *BC* of a ſmall angle *BAC* is divided into any number of equal parts *BH*, *HI*, *IC*, the lines, *HA*, *IA*, drawn from the points of diviſion to *A*, will divide the angle *BAC* into the ſame number of parts, which will be nearly equal among themſelves. For they would be ſo exactly if the line *BC* was an arch of a circle, drawn upon the center *A*; from which it differs ſo much the leſs as the angle at *A* is ſmaller; and ſo the propoſition is exacteſt in the ſmalleſt angles.

Small angles ſubtended by the ſame object are reciprocally as its diſtances from the eye.
[a] Art. 57.
[b] Art. 59.

60. When the diſtance *AB* is double or treble of *Ab*, the ſubtenſe *BC* will be double or treble of the ſubtenſe *bc* of the ſame angle at *A*[a]. Divide *BC* into its parts *BH*, *HI*, *IC*, each equal to *bc*, and the rays *HA*, *IA*, will divide the angle *BAC* into as many equal parts[b]. Therefore when two angles *bAc*, *BAH* are ſubtended by the ſame or by equal lines *bc*, *BH*, the magnitude of the firſt angle *bAc*, will be to the magnitude of the ſecond *BAH*, as the ſecond diſtance *BA* to the firſt diſtance *bA*.

II.
Experiment. To meaſure the focal diſtance of a globe of water, and of glaſs. Fig. 107.

61. Take an hollow globe of glaſs, or inſtead of it a thin round flaſk or decanter, and making a moderate round hole, about an inch broad, in a piece of brown paper, paſte it on one ſide of the belly of the decanter; and having filled it with water, hold the ſide that is covered, to the ſun, that the rays falling perpendicularly upon the hole, may paſs through the middle of the water; and the emergent rays will be collected to a focus, whoſe neareſt diſtance from the decanter will be equal to a ſemidiameter of the belly of it: as will appear by receiving the rays upon a paper held at that diſtance. That this effect is owing to the refraction of the water and not at all to that of the glaſs ſhell, appears to be reaſonable by the 37th article; and will appear to be fact, by trying the

experiment over again with the empty decanter. For the light paſſing through the hole, being received upon a paper, will be as broad as the hole it ſelf at all diſtances of the paper from the decanter. If the like experiment be tryed with a ſolid globe or ball of glaſs, the diſtance of its focus from the neareſt part of the ball will be one quarter of its diameter.

III. Experiment. To meaſure it after the firſt refraction only.

62. Things remaining as before, paſte a piece of thin white paper upon the ſide of the decanter oppoſite to the hole in the brown paper; and when the light of the ſun, which comes through the round hole, falls upon the white paper, meaſure its breadth, GH, with a pair of compaſſes; and it will be nearly equal to half the breadth, AB, of the hole in the brown paper. Which ſhews that if the converging rays AG, BH, were permitted to go ſtraight forward in a body of water continued far enough, they would convene at a focus, T, whoſe diſtance DT, from the neareſt point of the ball, would be about half[a] of, CT, its diſtance from the remoteſt point; and conſequently would be equal to the diameter CD; and therefore CT is to TE as 4 to 3, as was ſaid in the 33d article. If the white paper be paſted on the back ſide of a ſolid glaſs ball, the diameter GH of the circle of light will be found equal to one third part of AB; conſequently the rays AG, BH, converge to a focus T whoſe diſtance from D is one third[b] of its diſtance from C; that is, CT is to TD as 3 to 1 and conſequently CT is to TE as 3 to 2, as was ſaid in the 33d article. If the experiment be tryed with a lighted candle placed at a great diſtance; while the candle approaches to the ball, the breadth of the ſpot GH will increaſe continually; which ſhews that the focus T goes from the ball, and ſo it confirms the 34th article.

[a] Art. 57.

[b] Art. 57.

IV. Experiment. To meaſure the focal diſtance of a convex lens. Fig. 105.

63. Having covered either ſide of a convex glaſs with paper, in which there are ſeveral ſmall holes made with a pin, and having expoſed the glaſs directly to the ſun, the rays which paſs through the holes will appear like ſo many white ſpots upon a paper held pretty cloſe behind the glaſs; and theſe ſpots will come cloſer together as the paper is gradually drawn back from the glaſs till at laſt they all unite in one ſpot or focus. The diſtance of this focus from the glaſs may therefore be meaſured, and will not be ſenſibly altered by turning the other ſide of the glaſs to the ſun[c], nor by inclining it a little to the incident rays[d]; and provided this ſmall inclination be ſo made as not to move the middle point of the glaſs, the focus or ſpot upon the paper will not be ſenſibly moved. Which ſhews that the axis of the oblique pencil continues ſtraight as before[e]. If the paper be drawn farther from the glaſs, the ſpots will recede from each other.

[c] Art. 47.

[d] Art. 50.

[e] Art. 43.

V. Experiment. To meaſure the focal diſtance of a concave lens.

64. If a concave lens covered in like manner be expoſed to the ſun, the ſpots of light which come through the holes and fall upon the paper behind the glaſs, will continually recede from each other as the

paper is gradually removed from the glaſs. Which ſhews that the emergent rays continually diverge from a focus ſituated before the glaſs. Fig. 106. When the diſtance *ab*, of any two ſpots from each other, is double the diſtance *AB*, of the two correſponding holes in the cover, through which they came; the diſtance *Ef*, between the paper and the glaſs, is a Art. 57. then equal to its focal diſtance *EF*[a]; and by this means it may be meaſured.

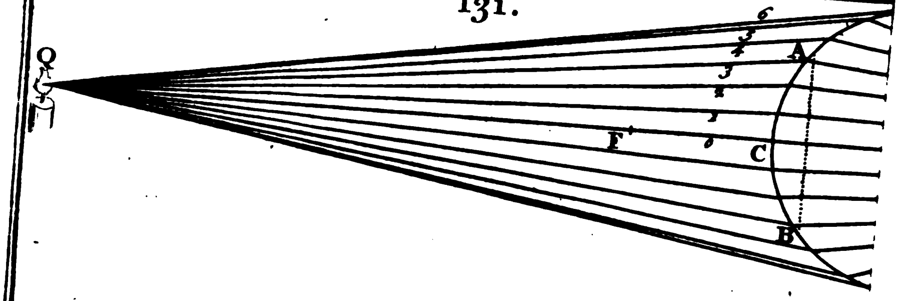
By theſe experiments it will be found, that the focal diſtance, *EF*, of a plano-convex or of a plano-concave glaſs is equal to a diameter of its convex or concave ſurface, that is, of the whole ſphere it belongs to: which proves the 33d article by holding the plane ſide of the glaſs perpendicular to the incident rays, that they may paſs through it unrefracted. Secondly, that the focal diſtance, *EF*, of a double-convex or double-concave glaſs, of equal convexities or concavities, is equal to a ſemidiameter of either of its ſurfaces: and conſequently that the focal diſtance of a glaſs of unequal convexities or unequal concavities will have an intermediate length between a diameter and a ſemidiameter of that ſurface which is moſt convex or moſt concave. For if a glaſs of equal convexities or concavities be conceived to grow gradually flatter on either ſide till it becomes quite flat, its focal diſtance will grow gradually b Art. 40, 41. longer[b], till at laſt it becomes a diameter of the remaining ſurface, as was ſaid above.

The like experiments may be tryed with a concave or a convex looking-glaſs covered with a paper ſtuck full of holes, to confirm the 26th article.

VI. Experiment. To ſhew the relation of conjugate focuſes of a lens to its principal focuſes. Fig. 128.

65. Having found the focal diſtance, *EF*, of a convex glaſs and fixed it flat againſt a moderate hole made in a thin board *CE*, placed upright upon a long table or floor; through the point *C*, directly under the middle of the glaſs, draw a long line *AB* perpendicular to the board; in which meaſure the focal diſtance of the glaſs from *C* to *F* and from *F* to I, I to II, II to III, &c; and alſo on the other ſide from *C* to *f* and from *f* to 1, 1 to 2, 2 to 3, &c; then taking $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the focal diſtance, ſet them off from *F* towards I and alſo from *f* towards 1, and put the figures $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, to the points of diviſion, as in the ſcheme; laſtly having darkened the room, if a candle be placed at *Q* over the mark I, the rays which paſs through the glaſs will be united at *q* upon a paper held over the oppoſite mark 1; and removing the candle to II and the paper to $\frac{1}{2}$, the rays will be united here alſo; and likewiſe when the candle and paper are removed to III and $\frac{1}{3}$, IV and $\frac{1}{4}$ &c: and the effect will be the ſame if the paper and candle be tranſpoſed into each others places: which confirms the 48th article. Beſides, it appears that *fq* varies reciprocally as *FQ* varies; that is, it decreaſes in the ſame proportion as *FQ* increaſes, and on the contrary.

66. Things



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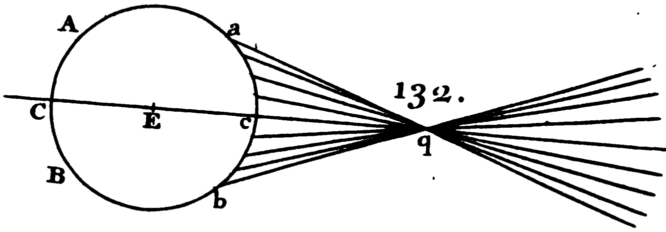

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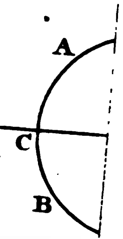

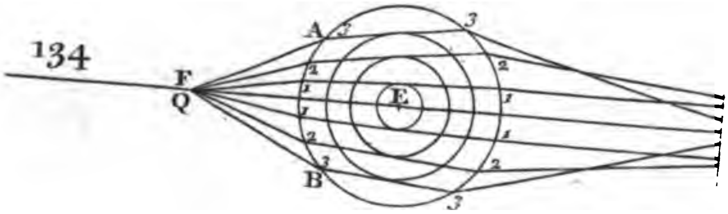

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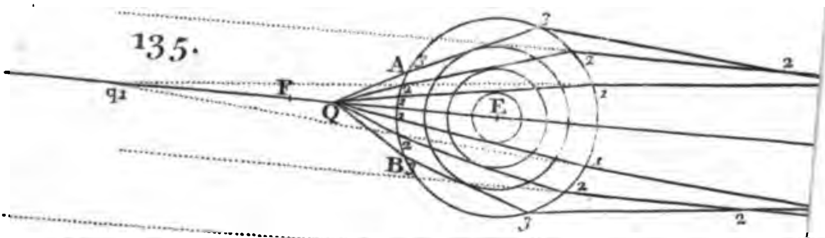

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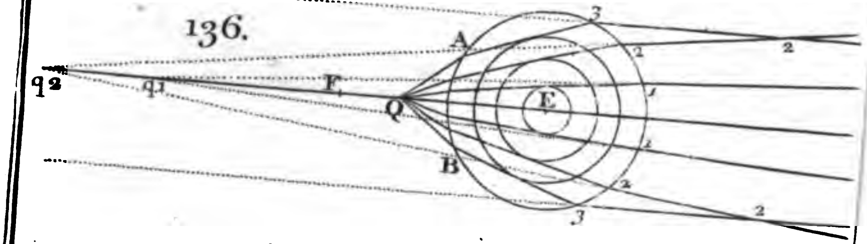

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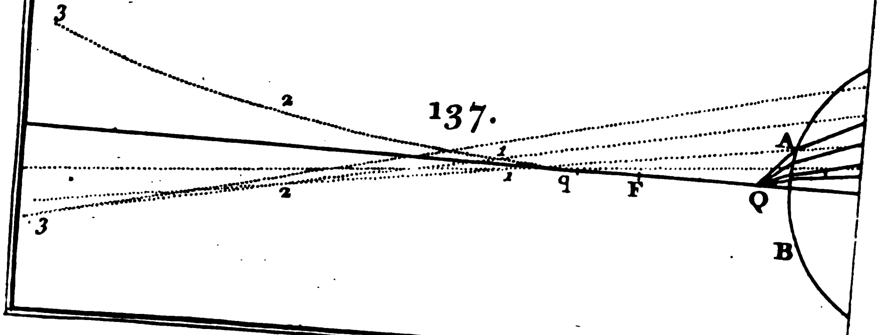


137.

VII.
Experiment.
To ſhew the inverſion and magnitude of an image.

66. Things remaining as they were, when a ſecond candle is placed on either ſide of the firſt at the ſame diſtance from the glaſs, the union of its rays will make another image upon the paper q, on the contrary ſide of the axis $\mathcal{Q}Eq$; and the diſtance between the two images will be found to bear the ſame proportion to the diſtance between the candles, as the diſtance of the images from the glaſs, bears to the diſtance of the candles from the glaſs. Theſe obſervations confirm the reaſon why the image of a ſingle candle is inverted upon the paper; and why its magnitude is altered when its place is altered. Becauſe what has been obſerved of two candles is applicable to any two points of the ſame candle; ſo that the deſcriptions of images in the 55th article is ſufficiently illuſtrated by this experiment. And what has been tryed with a convex lens may alſo be tryed with a concave looking-glaſs placed at the hole in the board.

VIII.
Experiment.
To make images of objects appear upright in a dark room.
Fig. 129.

67. If the rays of the ſun or moon or of a remote candle, which are made to converge to a focus q by a convex lens E, be intercepted by a looking-glaſs AB, they will be ſo reflected from it, as to converge to a focus $\mathcal{Q}$ at an equal diſtance before the looking-glaſs to that of q behind it. This may be examined by holding a piece of white paper at $\mathcal{Q}$ to receive the reflected rays. Therefore if the reflected rays be ſuppoſed to go directly backward, that is, from $\mathcal{Q}$ towards the looking-glaſs, AB, they will be reflected from it ſo as to diverge from q; which proves the 23d and 24th articles. If the convex-glaſs be placed in a hole of a window-ſhutter, and the room be darkened, the 130th figure ſhews how images of external objects, as $P\mathcal{Q}R$, which are ſeen inverted upon a paper held upright at pqr, may, by reflecting them downwards upon a paper held horizontally at $\varpi\kappa\varrho$, be viewed upright, when the ſpectator's back is turned towards the lens.

IX.
Experiment.
To ſhew the degrees of brightneſs and diſtinctneſs of an image.

68. Whatever be the ſhape and magnitude of the hole in the paper that covers part of a lens, the ſhape and magnitude of the picture of an object will be the ſame as when the lens is uncovered; becauſe any ſmall part of a pencil of rays has the ſame focus as the whole: but the brightneſs of the picture will be diminiſhed in proportion as the hole in the cover is diminiſhed; becauſe the quantity of light which illuminates every point of the picture is diminiſhed in that proportion. If the lens be very thick and broad, by this diminution of its aperture the diſtinctneſs of the picture may be ſenſibly improved. Becauſe the rays which fall upon the margin of ſuch a glaſs are not refracted exactly to the ſame points as thoſe which fall nearer the middle of it: which will be manifeſt by the next experiment.

X.
Experiment.
A cauſtick what and how formed by a globe or by a cylinder.
Fig. 131.

69. When the light of a candle or of the ſun is refracted through a globe, or through the belly of a round decanter filled with water, and falls upon a table cloth, or upon a piece of white paper held parallel and

very near to the axis of the light; the luminous figure there formed is bounded by two bright curves, called cauſticks; which in going from the globe approach towards one another and to the axis of the pencil, till they touch it and there make a ſharp angle, whoſe point is the focus of the pencil.

From the brightneſs of theſe curves it will appear, by inſpection of the figure, that they are formed by the ſucceſſive interſections of every ray with the next to it, taken in ſucceſſive order from one ſide of the globe to the other; and conſequently that the brightneſs of the paper within the curves, and its darkneſs without, is cauſed by a multitude of interſections of rays within, and by none at all without.

It will alſo appear by the figure and poſition of the cauſtick, that every ray croſſes the next ray in a point of the cauſtick before it cuts the axis. For if every ray croſſed the next ray in a point of the axis, they muſt all croſs it in one and the ſame point; and ſo the figure of the light upon the paper would conſiſt of two bright angular ſpaces, bounded not by curves but by ſtraight lines that croſs in the focus; and conſequently each angular ſpace, at equal diſtances on each ſide of the focus, would be equally bright; which is contrary to experience. Fig. 132.

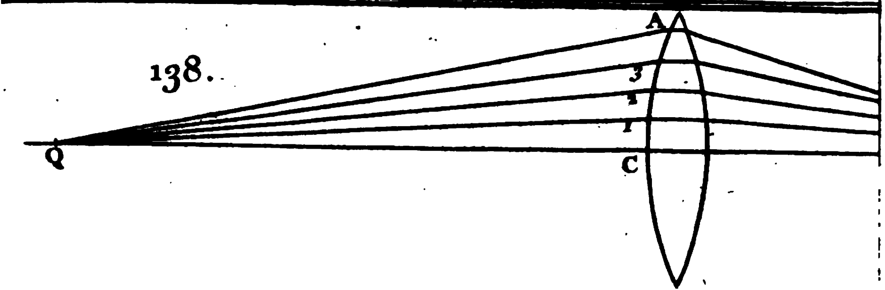
And if every ray croſſed the next ray, after it had cut the axis, their ſucceſſive interſections would form a bright curve, which would have a ſharp angle at the focus as before; but would diverge farther and farther from the axis in going from the globe, as repreſented in the figure; which is alſo contrary to experience. It is manifeſt then from the ſhape and poſition of the cauſtick, that every ray croſſes the next ray before it cuts the axis; and alſo that the focus of the pencil is that point of the axis where the neareſt rays cut it; and that the rays in the incident pencil, which lye farther and farther from the axis, cut it in ſeveral points, that lye farther and farther from the focus. Fig. 133. Fig 131.

An imaginary cauſtick formed by a globe or by a cylinder.

70. Therefore ſince the total bending of a ray is not altered, while it paſſes at equal diſtances from the center of the globe, and conſequently touches a circle concentrick to it, it follows, that while the candle is moved gradually towards the globe, the neareſt rays to its center, on oppoſite ſides of it, will firſt become parallel to the axis, and preſently after will diverge from a point behind the candle; then the next rays to theſe, on oppoſite ſides of the center, will alſo become parallel to the axis, and after that will diverge from another point of it, ſomewhat farther behind the candle than the former point, from which the neareſt rays diverged; and ſo on. Conſequently when the emergent rays diverge, each contiguous couple being produced backwards, will cut the axis before they croſs one another; and theſe ſucceſſive interſections, from whence each couple diverge, will form an imaginary cauſtick, beginning with a ſharp angle at the focus, and receding from the axis, in going backwards from the globe. Fig. 134. Fig. 135. Fig. 136. Fig. 137.

71. A

138.



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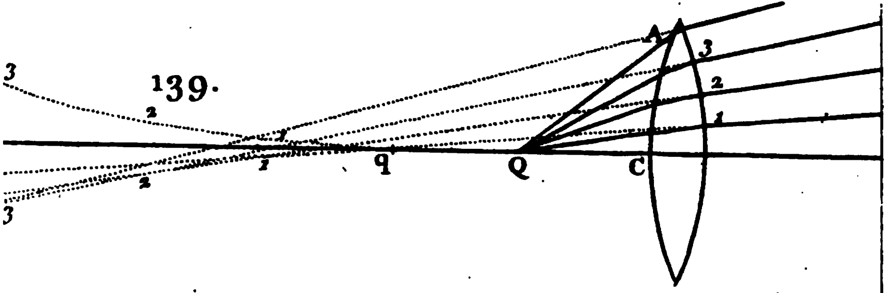

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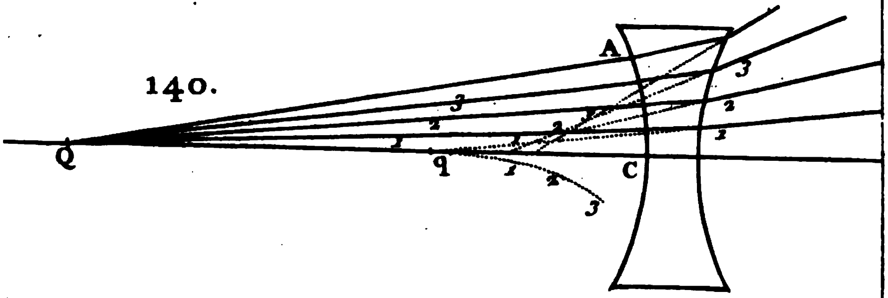

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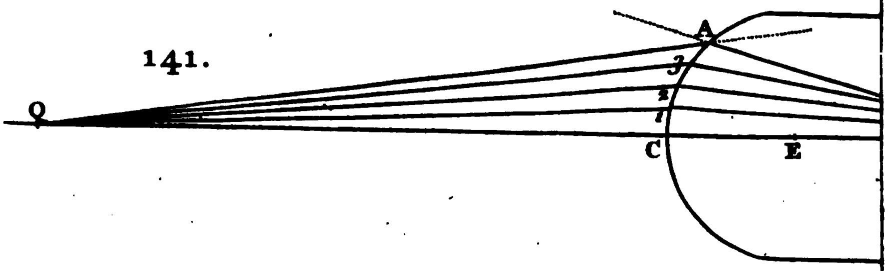

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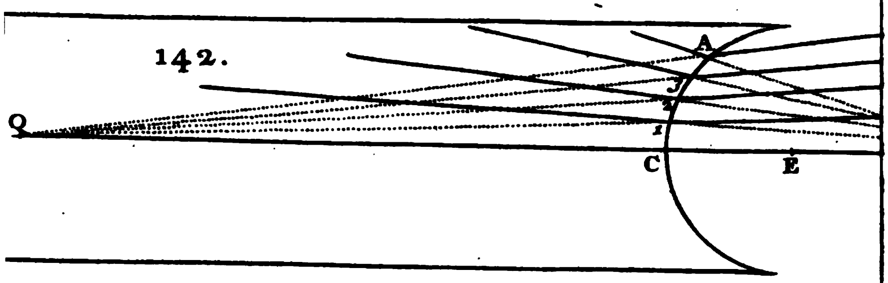

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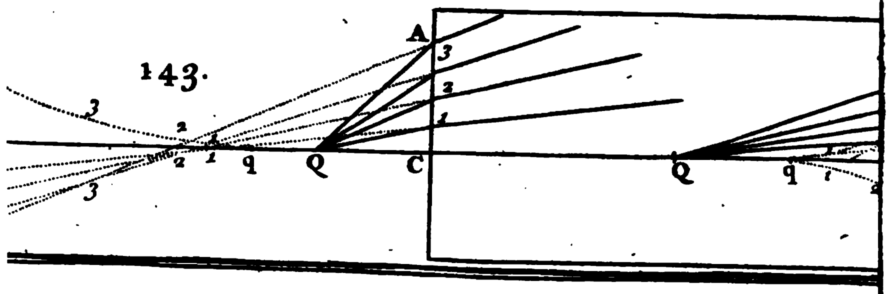

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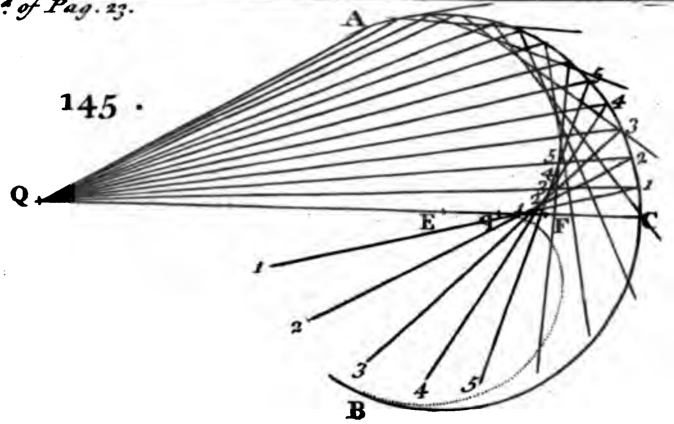

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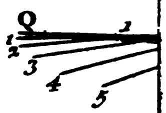

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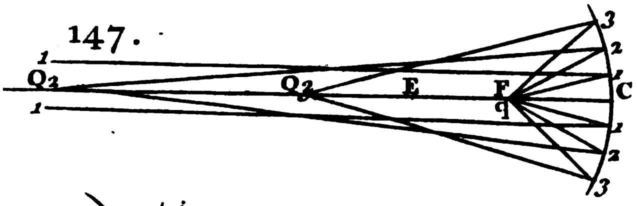

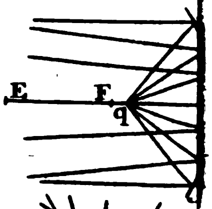

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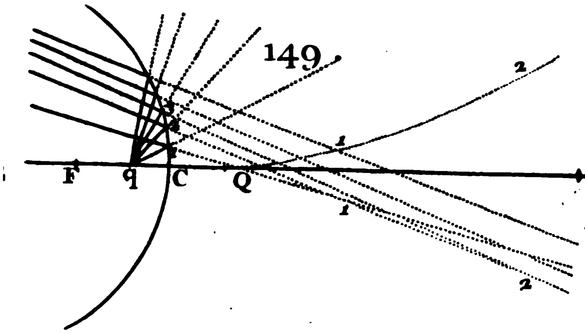

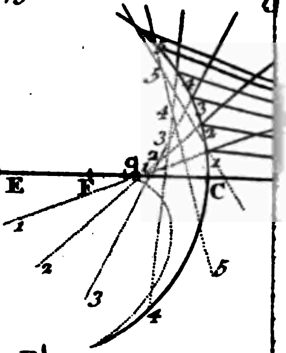

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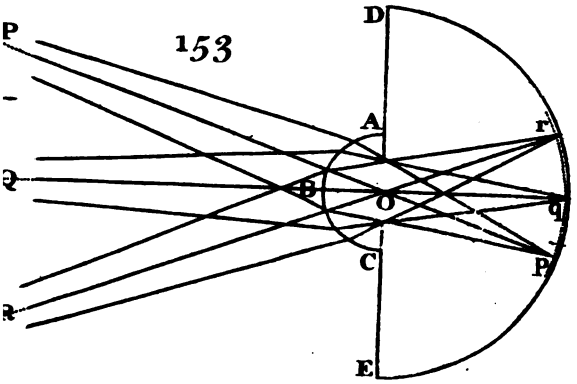

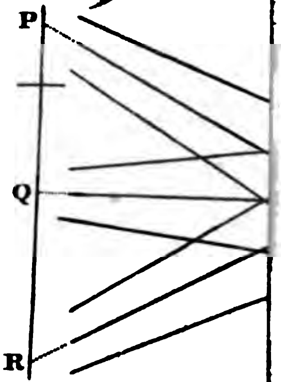

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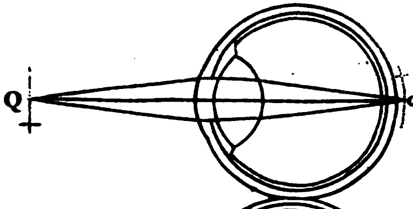

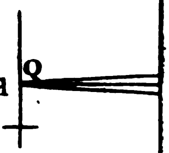

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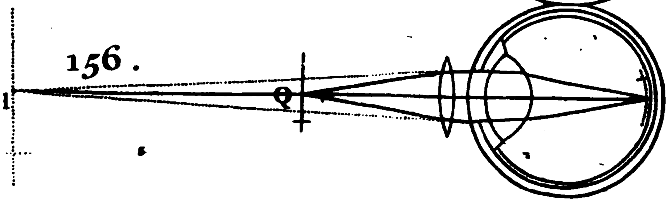

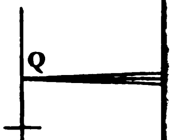

Cauſticks formed by a convex lens. Fig. 138, 139. Fig. 131.

71. A large pencil of rays refracted through a convex lens is alſo formed into a cauſtick, or ſome part of a cauſtick adjoining to the focus of the lens; and extending from it more or leſs according as the lens is compoſed of larger or leſſer ſegments of the ſpheres whoſe convexities it has. For conceiving two planes *AB*, *ab* to cut off two oppoſite ſegments *ACB*, *acb* from the globe, where the rays paſſed through it, their refractions through the ſegments, when joined together, will be much the ſame as when they were ſeparated by the middle part of the globe that lay between them; and conſequently the cauſticks formed by the lens and by the globe will have like properties.

The truth of this deſcription of cauſticks, may be farther confirmed by covering one ſide of the globe or of a broad convex lens, with a large circle of brown paper, whoſe diameter has a row of pin-holes made in it at any equal diſtances from one another. For the ſpots of light which came through theſe holes, will appear upon a white paper, at equal diſtances from one another, when the paper is held perpendicular to the rays and near to the glaſs. But while it is withdrawn from the glaſs, the intervals between the exteriour ſpots will grow leſs than the intervals between the interiour, and will ſooner unite.

And by a concave lens. Fig. 140.

72. On the contrary if the ſame cover be put upon a concave lens, while the paper is drawn from the lens, the intervals between the exteriour ſpots will grow larger than the intervals between the interiour: which ſhews that the exteriour rays diverge from points that are nearer the concave, than thoſe from whence the interiour rays diverge. But this experiment will not ſucceed with common concave glaſſes made for ſhort-ſighted perſons: they are not concave enough, nor broad enough, nor thick enough to make this effect ſenſible.

The bendings of the exteriour rays too great.

73. It appears by theſe real and imaginary cauſticks, that the exteriour rays of a pencil are gradually too much bent; or, which is the ſame thing, the interiour rays are gradually too little bent, to belong all together to a ſingle point after refraction; and conſequently the angles of incidence of the exteriour rays are too large for that purpoſe, both at the firſt and ſecond ſurface of the globe or lens.

Cauſticks formed by refraction at a ſingle ſpherical ſurface. Fig. 141, 142.

74. Conſequently the like cauſticks muſt be formed by the refractions of a pencil of rays at a ſingle ſurface; only theſe cauſticks will approach ſlower towards the axis than the former, or recede from it ſlower; becauſe every couple of contiguous rays has now but a ſingle refraction to make them converge, or but a ſingle refraction to make them diverge.

And by a plane ſurface. Fig. 143, 144.

75. And it is demonſtrated in the next book, that the rays of a large pencil, refracted by a ſingle plane ſurface, will alſo diverge from the points of an imaginary cauſtick; which begins at their focus and recedes from the ſurface, when the refractions are made out of a rarer into a denſer medium; and accedes towards it, when they are made out of a denſer into a rarer.

A cauſtick formed by reflection at a ſpherical or cylindrical concave ſurface.

76. The 145th figure repreſents the ſhape of a cauſtick formed by the ſucceſſive interſections of the contiguous rays of a large pencil, reflected from a concave ſurface, either ſpherical or cylindrical. Such cauſticks may be ſeen upon the ſurface of milk, or upon any opake whitiſh mixture of liquors contained in a white china-cup, or upon the bottom of a ſnuff-box, whoſe rim is well poliſhed, when the light of a candle or of the ſun or of a remote window ſhines upon it.

Another cauſtick formed beyond the center of the concave. Fig. 145, 146.

77. While the points of incidence keep fixt, imagine all the lines deſcribed by the reflected rays to accede towards the center, till they unite at the focus of the pencil; and ſuppoſing the rays to go backwards in the ſame lines, after this ſecond reflection they will all recede from their former focus and accede towards the contrary ſide of the center [a]: and the exteriour rays, whoſe firſt interſections with the axis were fartheſt from the center, will now come neareſt to the oppoſite ſide of it. So that when the luminous point is placed between the principal focus and the center, another cauſtick will be formed beyond the center.

[a] Art. 27.

An imaginary cauſtick by reflection from the concave. Fig. 146, 147, 148, 149.

78. Therefore while this luminous point is moved gradually towards the ſurface, when it comes to the principal focus, the rays that are neareſt to the axis will firſt become parallel to it, and preſently after will diverge from a point behind the concave; then the next rays to theſe will alſo become parallel to the axis, and after that will diverge from another point of it, ſomewhat farther behind the ſurface than the former, from which the neareſt rays diverged. Conſequently each couple of reflected rays, which lye contiguous, being produced backwards, will cut the axis before they meet one another. And theſe ſucceſſive interſections, from whence each couple diverge, will form an imaginary cauſtick behind the concave, beginning with a ſharp angle at the focus and receding from the axis in going from the ſurface.

An imaginary cauſtick by reflection from the convex ſide of the ſurface.

79. While the points of incidence keep fixt imagine all the interſections at the axis in the 149th figure to be ſhoved to the focus Q, as in the 150th figure, and let rays diverge from it upon the convex ſide of the ſurface; and the reflected rays produced will all ſeparate from the former focus *q*, ſo as to form an imaginary cauſtick, and the remoteſt rays from the axis, whoſe interſections were ſhoved the fartheſt towards the ſurface, will accede the neareſt to it after reflection [b].

[b] Art. 27.

A general obſervation upon cauſticks.

80. In all theſe cauſticks by refraction and reflection at plane and ſpherical ſurfaces, the concourſe of two contiguous rays (produced) lyes farther from the focus, and from the axis, according as their points of incidence are farther from the axis. It is obſervable that a pencil of rays reflected from a plane ſurface will form no cauſtick at all, becauſe they diverge accurately from a ſingle point [c].

[c] Art. 23.

No ſpherical figure can refract or reflect

81. From what has been ſaid it appears that a ſpherical ſurface, having every where the ſame degree of curvity, can neither reflect nor re-

fract all the rays of a large pencil to a ſingle point; and that a ſingle ſurface fit for this purpoſe muſt grow ſtraighter or flatter gradually in going from its axis[a], as repreſented in the 151ſt figure; and that if one ſide of a lens be ſpherical, it is not ſufficient that the other be plane, but it muſt be convex in the middle, to ſhorten the focal diſtance of the middlemoſt rays, and muſt become concave towards its circumference, to prolong the concurrence of the outermoſt rays; as repreſented in the 152d figure. Nevertheleſs the middlemoſt rays of a pencil are crowded ſo cloſe together by reflection and refraction at ſpherical ſurfaces and lenſes, and the outermoſt rays are ſcattered ſo thin upon a plane paſſing through the focus perpendicular to the axis, that the confuſion they make in a picture, by mixing with rays of other pencils, is ſeldom ſenſible when the glaſs has a moderate aperture. And ſince the unequal degrees of refrangibility of rays of different colours (to be explained in the 6th chapter) makes far greater aberrations from the focus, than thoſe that are made by the ſphericalneſs of the figure, it would be to little purpoſe to give glaſſes any other figure than ſpherical; eſpecially conſidering the great difficulty there would be in the mechanical operation.

rays to a ſingle point.

[a] Art. 73, 76.

CHAP. III.

CONCERNING THE EYE AND MANNER OF VISION.

82. CONSIDERING what has been ſaid in the 33d and 35th articles, one might contrive a tolerable eye in this manner, by placing a pellucid hemiſphere *ABC* to ſerve for the fore part, and another concentrick one *DqE*, oppoſite to the former, to ſerve for its bottom or back part; making the ſemidiameter, *Oq*, of the latter triple the ſemidiameter, *OB*, of the former; and then by filling the whole cavity of both with water. By this means rays of light flowing from the points P, Q, *R*, &c, of remote objects, after refraction at the ſurface *ABC*, will be collected to as many other points *p*, *q*, *r*, of the cavity *DqE*, and paint an image upon it. And becauſe a ſpherical ſurface does not accurately refract all the rays of a large pencil to a ſingle point[b], but only thoſe that go pretty near its axis; this imperfection might be remedied by covering the baſe *AC*, of the leſſer hemiſphere, all but a moderate hole about the center *O*; which would anſwer the purpoſe much better than if the ſurface it ſelf was covered, all but a hole in the middle about *B*. For in this latter caſe the ſurface *ABC* would not receive rays from the lateral points *P*, *R*, ſo directly as thoſe from the middle of the object, to all which it is expoſed alike when the hole is left open at the center *O*.

A fictitious eye deſcribed by *Hugens*[*]. Fig. 153.

[b] Art. 81.

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[*] Opuſcula Poſtuma. p. 112.

And compared to the natural eye.

83. Though this conſtruction of the eye appears not amiſs at firſt ſight, yet we ſhall ſee preſently that the author of nature has wiſely varied ſome things for the better, and added others abſolutely neceſſary; though in every thing we cannot perceive his deſigns. In the firſt place he would not make uſe of an intire hemiſphere ABC, but retaining the middle part, has taken off pretty much from the ſides, and yet without contracting the compaſs of objects taken in at one view. The reaſon of this was to bend inwards the edges of the larger hemiſphere about D and E, thereby reducing the ſhape of the eye to a rounder figure, for the convenience of its motion every way in the cavity that contains it. He has therefore given it ſuch a ſhape, as is expreſſed in this other figure, repreſenting an human eye diſſected through its axis, all the parts being twice as big as in the life to render them more conſpicuous.

Fig. 154.

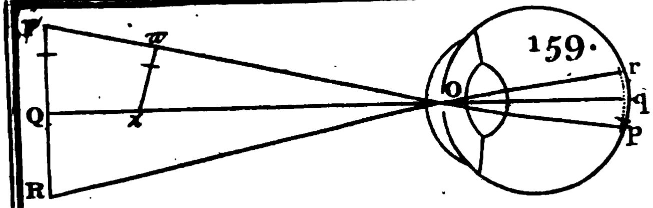
An human eye deſcribed.

84. Here the tranſparent parts of the coat called the cornea is ABC; the remainder $ATYC$ being opake, and a portion of a larger ſphere. Within this outward coat anatomiſts diſtinguiſh two others; the innermoſt of which is called the retina, being like a fine net compoſed of the fibres of the optick nerve YVT woven together, and is white about the parts p, q, r, at the bottom of the eye. The cavity of the eye is not filled with one liquor, but with three of different ſorts. That contained in the outward ſpace $ABCOEGFDO$ is called the aqueous humor, being perfectly fluid like water; the other contained in the inward ſpace $EpqrDFG$ is a little thicker like the white of an egg, and is called the vitreous humor; the third humour FG is ſhaped like a lens of unequal convexities, lying between the two former, and fixed to the ſide coats by filaments or threads extended all round it, and is called the cryſtalline humor, being hard like the white of an egg boiled, but as clear as the other two, and differs from them in a greater degree of refractive power; whereby the rays that came from the points P, Q, R, having received a degree of convergence by the refraction of the cornea ABC, are made to converge a little more by other refractions at the ſurfaces of the cryſtalline FG; ſo that uniting in as many other points p, q, r, upon the retina, they repreſent the points of the object P, Q, R from whence they came. And perhaps the rays are ſo directed by theſe ſecondary refractions at the cryſtalline, as to fit the cavity pqr intended to receive them; which otherwiſe muſt have been a portion of a larger ſphere[a], according to the fictitious deſign in the former figure.

[a] Art. 73, 80.

The cryſtalline makes all pictures diſtinct.

85. Beſides this there was greater need of the lens FG upon another account; namely to help the eye to conform it ſelf for ſeeing objects diſtinctly at all diſtances, which was wanting in the fictitious eye. There are two ways of doing it by the help of this lens FG, in order to ſee things near at hand; either by moving it nearer to the outward cornea, or by increaſing its convexity, or perhaps by doing both at once. If it is



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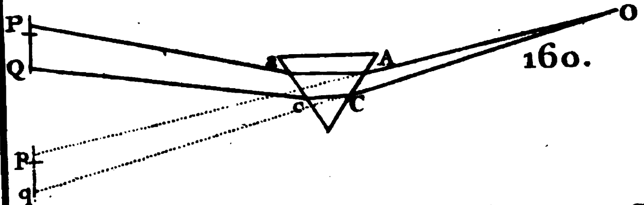

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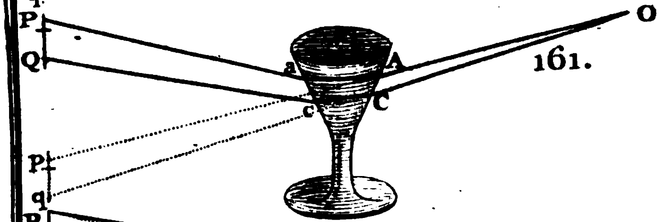

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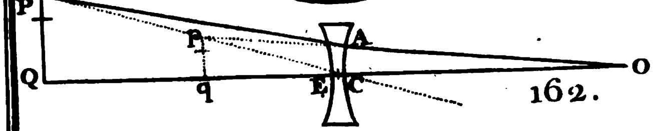

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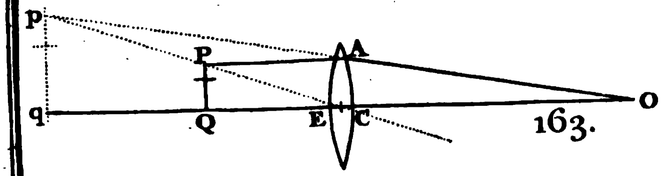

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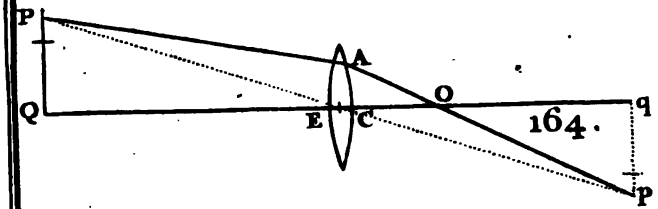

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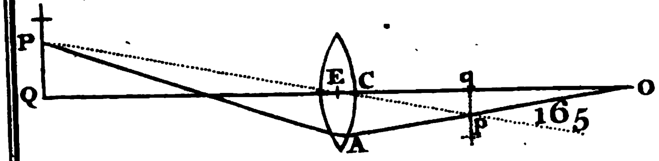

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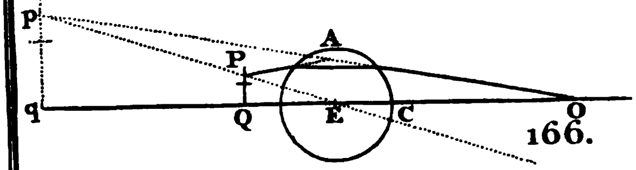

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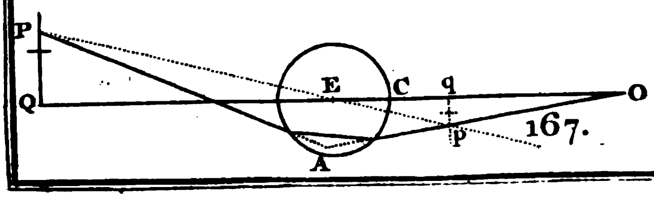

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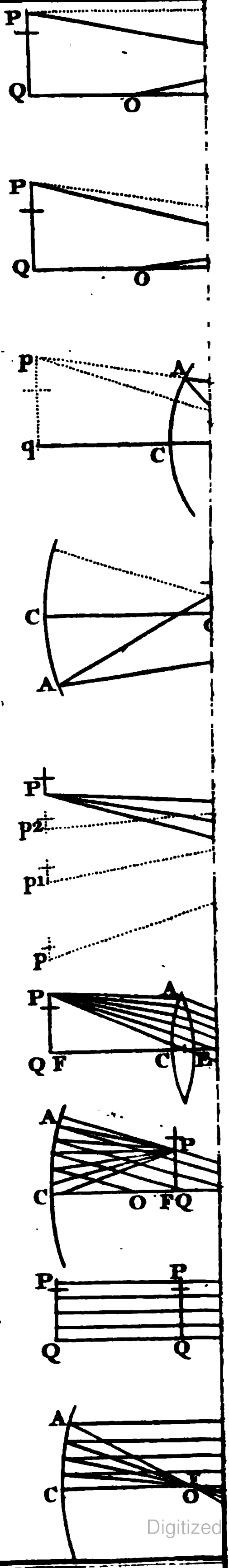

is moved towards the cornea, this may be effected by the preſſure of the muſcles againſt the ſides of the eye, and conſequently againſt the vitreous humor; but if the cryſtalline alters its figure and becomes rounder for ſeeing near objects, the filaments DF, EG, whoſe greater tenſion helps to flatten it, may perhaps be ſlackened by the lateral preſſure aforeſaid; and poſſibly both theſe alterations are made at the ſame time. The hole or pupil O is not placed in the center of the cornea ABC, as in the fictitious eye, but ſomewhat nearer to its front. The reaſon is uncertain, unleſs this alſo may contribute to make the images coincide with the cavity of the retina, (in all their parts,) which otherwiſe muſt have been ſhaped according to a larger ſphere [a].

Some dimenſions of an human eye.

86. The diameter AY of the ſphere of the eye is about an inch of the Rheinland foot, which is much the ſame as the old Roman foot: and the diameter of the outward cornea is about three fifths of an inch; the breadth of the pupil, O, has no fixt meaſure, being greater or ſmaller, as any one may try, according as leſs or more light ſhines upon the eye; it alſo contracts at the near approach of any ſmall object, when we endeavour to view it diſtinctly. Its fabrick is admirable in this reſpect that while it alters its ſize it preſerves its roundneſs [1]. So far Mr. *Hugens* [2], to which I add ſomething from Sir *Iſaac Newton* [3].

Pictures on the retina the cauſe of viſion.

87. This account of the eye and of the cauſe of viſion is farther confirmed by theſe arguments; that anatomiſts when they have taken off from the bottom of the eye that outward and thickeſt coat called the dura mater, can ſee through the thinner coats the pictures of objects lively painted thereon. And theſe pictures propagated by motion along the fibres of the optick nerves into the brain, are the cauſe of viſion. For according as theſe pictures are perfect or imperfect, the object is ſeen perfectly or imperfectly. If the eye be tinged with any colour (as in the diſeaſe of the jaundice) ſo as to tinge the pictures in the bottom of the eye with that colour, then all objects appear tinged with the ſame colour.

Confuſed pictures in old mens eyes how cauſed, and mended by convex glaſſes. Fig. 155.

88. If the humours of the eye decay by old age ſo as by ſhrinking to make the cornea and coat of the cryſtalline humour grow flatter than before, the light will not be refracted enough, and for want of a ſufficient refraction will not converge to the bottom of the eye, but to ſome place beyond it; and by conſequence will paint in the bottom of the eye a confuſed picture; and according to the indiſtinctneſs of the picture the object will appear confuſed. This is the reaſon of the decay of ſight in old men, and ſhews why their ſight is mended by ſpectacles. For the convex glaſſes ſupply the defect of plumpneſs in the eye, and by increaſing the refractions make the rays converge ſooner, ſo as to convene di-

Fig. 156.

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[a] Art. 73, 80.

[1] See this accounted for in Mr. *Cheſſelden*'s anatomy p. 319. 3d. Ed.

[2] Dioptrica prop. 31.

[3] Opticks pag. 12.

ſtinctly at the bottom of the eye, if the glaſs has a due degree of convexity.

Confuſed pictures in ſhort-ſighted eyes, how cauſed and mended by concave glaſſes.
Fig. 157.
a Art. 34, 48.
Fig. 158.

89. And the contrary happens in ſhort-ſighted men, whoſe eyes are too plump. For the refraction being now too great, the rays converge and convene in theſe eyes before they come at the bottom; and therefore the picture made in the bottom and the viſion cauſed thereby will not be diſtinct, unleſs the object be brought ſo near the eye as that the place where the converging rays convene may be removed to the bottom [a]; or that the plumpneſs of the eye be taken off and the refraction diminiſhed by a concave glaſs, of a due degree of concavity; or laſtly that by age the eye grows flatter till it comes to a due figure. For ſhort-ſighted men ſee remote objects beſt in old age, and therefore they are accounted to have the moſt laſting eyes. So far Sir *Iſaac Newton*.

The pupil of the eye may be conſidered as a point.
Fig. 153.
b Art. 91.
c Art. 97.

90. In determining the magnitude of pictures upon the retina, only one ray in each pencil need be conſidered; becauſe when the picture is diſtinct, all the rays in any one pencil are collected to one and the ſame point of the retina. Or, which is much the ſame, we may ſuppoſe the pupil of the eye contracted to a point: and, for greater ſimplicity and eaſe of the imagination that this point O is a little hole at the center of a dark, hollow hemiſphere DqE, admitting only ſingle rays ſtraight through it without any refraction at all. For then the lengths of theſe pictures pqr will increaſe and decreaſe as the angle pOr does, or as the angle POR does; which I am going to ſhew to be the property of the natural eye [b]: and if the ſemidiameter Oq, of this hollow hemiſphere, be about $\frac{5}{8}$ of an inch, or of the axis of an human eye, the pictures of the ſame objects will always have the ſame bigneſs in both ſorts of eyes [c], very nearly.

Diameters of pictures on the retina are as the angles ſubtended by the object at the eye.
Fig. 159.

91. The diameters or lengths of the pictures of objects upon the retina are meaſured by, or proportionable to, the angles which the rays that come from the extremities of the object do make in falling on the eye; provided thoſe angles be but ſmall. For let two or more objects PQ and $\varpi\varkappa$, either parallel or oblique to each other, ſubtend the ſame angle POQ or $\varpi O\varkappa$ at O; and becauſe the particles of light flowing from P and ϖ deſcribe the ſame line $P\varpi O$, they will be refracted to the ſame point p upon the retina; and in like manner thoſe that flow from Q and $\varkappa$ will be refracted to the ſame point q; and ſo the pictures pq of the objects PQ, $\varpi\varkappa$, which ſubtend the ſame angle at O, are the ſame in magnitude; which was the firſt thing to be proved.

d Art. 87.

Now the pictures of objects painted upon the retina of a dead eye are found by experience to be perfectly well ſhaped and proportioned in their parts [d]; that is, the proportion of the parts pq, qr, of the whole picture pqr, is the ſame as that of the parts PQ, QR, of the whole object PQR; and this latter proportion is very nearly the ſame as that of the angles POQ, QOR ſubtended by the parts PQ, QR[a]; and ſo the propoſition is proved when the objects PQ, QR are both at the ſame diſtance from the eye. And ſince it was ſhewn juſt before, that the objects PQ and ϖx have the ſame picture pq, it follows that the proportion of the pictures of the objects ϖx and QR is the ſame as that of the angles ϖOx, QOR, ſubtended by them at the eye.

Are reciprocally as the diſtances of the object from the eye.

92. When an object approaches towards the eye, the diameter of its picture upon the retina increaſes in the ſame proportion as the diſtance between the eye and the object decreaſes; and on the contrary, it decreaſes in the ſame proportion as that diſtance increaſes. For the diameter of its picture increaſes in the ſame proportion as the angle increaſes, which the object ſubtends at the eye[b]; and this angle, when ſmall, increaſes in the ſame proportion as the diſtance between the eye and object decreaſes[c].

Brightneſs of pictures not altered by the diſtance of the object from the eye.

93. The degree of brightneſs of the picture of an object painted upon the retina continues the ſame, at all diſtances between the eye and the object; provided none of the rays be ſtopt by the way, and that the pupil does not alter its aperture. For inſtance, when the eye approaches as near again to the object, the picture upon the retina becomes double in length and double in breadth, and conſequently quadruple in ſurface; for the ſurface would be double, if its length alone or breadth alone was double. The quantity of rays received through the ſame aperture of the pupil, at half the diſtance from the object, is alſo quadruple[d]; and being equally ſpread over four times the quantity of ſurface of the retina, they are juſt as denſe as before when the object was at twice the diſtance.

Faintneſs of pictures of remote objects how cauſed.

94. It follows then that the faint appearance of remote objects is owing to the opacity of the atmoſphere, which hinders part of their light from coming to the eye. Accordingly we find that the ſun, moon and ſtars appear very faint when near the horizon, and brighter continually as they riſe higher; becauſe the tract of vapours, which lies in the way of the rays, is longeſt and thickeſt near the horizon; and becomes thinner and ſhorter as the objects riſe higher, and conſequently does leſs obſtruct the paſſage of the rays.

Their degrees of brightneſs by day-light and moon-light compared.

95. The ſenſibility of the eye, or its power to diſcern objects, without inconvenience, by different quantities of light, is vaſtly extenſive. For inſtance, the diſproportion in the quantities of light, caſt upon the horizon by the ſun and moon, at any equal altitudes, I find is no leſs than 90 thouſand to 1, when the moon is full; or no leſs than 180 thouſand to 1, when the moon is in the quarters. And the proportion between thoſe parts of the lights of the ſun and moon, whatever they be, which are reflected to our eyes from the ſame object by day and by night, can hardly be different from the proportion of the whole lights. Allowing

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[a] Art. 59.

[b] Art. 91.

[c] Art. 60.

[d] Art. 58.

then that the aperture of the pupil may poſſibly be 8 or 9 times leſs by day than by night, (that is about 3 times leſs in diameter,) yet the proportion in the quantities of day-light and moon-light, received by the eye from the ſame object, to illuminate a picture of the ſame bigneſs, will be no leſs than 20 thouſand to 1, when the nights have a middle degree of moon-light; I ſay no leſs, becauſe the numbers here given are deduced from a rule, which is built upon this principle; that the moon reflects all the light received from the ſun; which cannot be true, by reaſon of the appearance of very large obſcure places in her body; and in all probability a great part of the incident light is buried and loſt even in the brighteſt places.

The rule I mentioned is this, day-light is to moon-light as the ſurface of an hemiſphere, whoſe center is at the eye, to the part of that ſurface which appears to be poſſeſſed by the enlightened part of the moon: ſo that the whole heavens covered with moons would only make day-light. This will be evident enough from the following conſiderations, though I invented it another way. Day-light is made by innumerable reflections of the ſun's rays from all ſorts of bodies till at laſt they come to our eyes: for if this were not ſo, we could ſee nothing in the world, even in the day time, but the ſun and ſtars and ſelf-ſhining ſubſtances[a]. Accordingly we find that day-light is much the ſame, whether the ſun ſhines out or not, in the place we are in; becauſe his light is reflected to us from a vaſt quantity of earth, air and clouds extended all round us, perhaps to a hundred miles or more. So that the abſence of the ſun's rays from a particular place ſcarce alters day-light. Another thing is that the moon by day appears like a cloud in the air of a middle degree of brightneſs; ſome appearing duller and ſome brighter than the moon it ſelf. The rays of the ſun being therefore intercepted in the night from all the viſible clouds, and being reflected to us by the moon only, it follows that day-light is to moon-light, as the apparent ſurfaces of all the viſible clouds, to the apparent ſurface of the viſible part of the moon, conſidered as the only cloud which remains enlightened. And theſe two lights, whatever be the diſtances of the moon and clouds, are juſt the ſame as if thoſe bodies were all placed at any equal diſtances from us, and compoſed the ſurface of an hemiſphere[b]; whoſe parts are the true meaſures of the parts of the light which comes to us.

[a] Art. 2.

[b] Art. 93.

And confirmed by experiments with burning-glaſſes.

96. A vaſt diſproportion between the lights of the ſun and moon appears alſo by experiments made with burning-glaſſes; either by refraction of the rays through very broad lenſes, or by reflection from very broad concave-glaſſes or metals: which by collecting the rays of the ſun into a ſmall round image at the focus, do excite a more violent heat and burn quicker than the hotteſt wind-furnaces: as appears by their melting and calcining the hardeſt metals, and by vitrifying bricks and ſtones,

in much leſs portions of time than a minute [1]. Yet the rays of the moon being collected by the ſame glaſſes, do not excite the leaſt ſenſible heat; nor do they ſenſibly affect the niceſt thermometer, when caſt upon the ball of it [2], though the brightneſs of the light be very ſenſibly increaſed. By meaſuring the breadth of the round image at the focus, and by comparing it with the breadth of the glaſs it ſelf, it appears that ſome of theſe burning-glaſſes collect the incident rays into a ſpace about 2 thouſand times leſs than they poſſeſſed at their incidence. But by the preceding calculation, the light of the full moon muſt be condenſed about 90 thouſand times[a], to make it as denſe and as warm as the direct rays of the ſun. It is no wonder then that the heat of the moon's rays is not ſenſible in the focus of the glaſs, being then even 40 or 50 times thinner than the direct rays of the ſun. For it is found by experiments made with theſe glaſſes that the degrees of heat are proportionable to the denſities of the rays: which being compared with a ſcale of the degrees of heat and warmth of ſeveral natural bodies, determined by Sir *Iſaac Newton*, in the philoſophical tranſactions [3], it appears there is a vaſt diſproportion between the degrees of light which the eye can bear and be ſenſible of, and the degrees of its heat which the touch can bear and be ſenſible of.

a Art. 95.

Viſion limited both by magnitude and diſtance.

97. Dr. *Hook* aſſures us that the ſharpeſt eye cannot well diſtinguiſh any diſtance in the heavens, ſuppoſe a ſpot of the moon's body, or the diſtance of two ſtars, which ſubtends a leſs angle at the eye than half a minute; and that hardly one of a hundred can diſtinguiſh it when it ſubtends a minute [4]. If the angle be not bigger than this, the two ſtars appear to the naked eye, as if they were but one. I have been preſent at making the experiment, when a friend of mine, who had the beſt eyes of all the company, could ſcarce perceive a white circle upon a black ground, or a black circle upon a white ground, or againſt the ſky-light, when it ſubtended a leſs angle at the eye than two thirds of a minute; or which is the ſame thing, when its diſtance from the eye exceeded 5156 times its own diameter: which agrees well enough with Dr. *Hook*'s obſervation. Hence I find, by a rule in the next book, that the diameter of the picture of that circle upon the retina was but the 8000th part of an inch at moſt; and this may be called a ſenſible point of the retina. That this point is very ſmall any one may perceive from hence that the breadth of the fineſt hair is viſible at the length of ones arm.

Apparent magnitude to the naked eye defined.

98. The apparent magnitude of an object is a quantity of viſible extenſion, meaſured by, or proportionable to, the angle which the two [b] rays

b Art. 90.

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[1] Phil. Tranſ. abr. by *Lowth*. Vol. 1. p. 211. and by *Jones* Vol 4. p. 190.

[2] Ibid. *Lowth*. p. 213. and Mem. de l'Acad. Roy. des Scien. ann. 1705. p. 455. 8°.

[3] N°. 270. or *Jones*'s abr. Vol. 4. part. 2. p. 1. It. Mem. de l'Acad. an. 1703. p. 233.

[4] Animadverſions on Hevelii machina cœleſtis p. 8.

that come from the extremities of the object do make in falling on the eye. For the extremities of the object are ſeen in the directions of theſe rays; and in proportion as they make a greater or ſmaller angle at the eye, the magnitude of the picture upon the retina is longer or ſhorter[a]; and conſequently cauſes a ſenſation of a greater or ſmaller viſible extenſion; conſiſting of a greater or ſmaller number of viſible points, anſwering to the number of ſenſible points of the retina[b], of what magnitude ſoever theſe points are ſuppoſed to be.

[a] Art. 91.

[b] Art. 97.

How it varies.

99. The apparent magnitude of any given object is reciprocally as its diſtance from the eye: that is to ſay when the object approaches to the eye, its apparent magnitude increaſes (in proportion) as its real diſtance decreaſes; and on the contrary, it decreaſes (in proportion) as that diſtance increaſes. For the apparent magnitude of an object was defined to be a quantity of viſible extenſion proportionable to the angle which the object ſubtends at the eye[c], and this angle increaſes very nearly in proportion as the real diſtance between the eye and object decreaſes[d].

[c] Art. 98.

[d] Art. 60.

When called true magnitude.

100. The apparent magnitude of an object ſeen by the naked eye, in oppoſition to its apparent magnitude ſeen through glaſſes, and for ſhortneſs of expreſſion, is often called its true magnitude. And in ſpeaking of the apparent magnitude of an object I always mean the apparent magnitude of its diameter, or of its length or breadth, or of any principal line of it, and not of its ſurface or ſolidity, unleſs it be particularly ſpecified.

CHAP. IV.

CONCERNING VISION WITH GLASSES.

Apparent directions of viſible points defined.

101. ANY ſmall object or point of an object, ſeen by refracted or reflected rays, appears ſomewhere in the direction of that line, which the viſual ray deſcribes after its laſt refraction or reflection in falling upon the eye.

In the experiments to prove the laws of reflection and refraction[e], the pin at *B*, ſeen by a ray reflected from the water, appeared ſomewhere in the line *AC* produced, which the viſual ray *BCA* deſcribed after reflection at *C*, when it advanced to the eye. And as the whole line *CE* appeared lifted up by the refraction at the water, as if it had been a continuation of the line *AC* ſtraight on, ſo if a ſtraight oar be in part immerſed obliquely in water, it appears crooked, as if the part immerſed was broken at the ſurface, and lifted higher. For this part of the oar is ſeen in the direction of rays which are bent downwards by refraction at their emergence from the water, and conſequently advance to the eye as if they came from a place in the water which is higher than the real place of the oar. In like manner any point *P* of an object ſeen by the

[e] Art. 18. Fig. 4.

the ray *PAO* twice refracted, either by paſſing through the edge of a prism, or of a concave or convex lens, or through the ſides of a globe or decanter, or of a drinking glaſs filled with any tranſparent liquor; or ſeen by a ray *PAO* reflected from a plane or ſpherical looking-glaſs, appears to the eye at *O*, ſomewhere in the direction of the laſt refracted or reflected ray *AO*. Laſtly an object *P* viewed by the eye at *O*, through a multiplying glaſs, appears at one view in as many different places, *p*, *p* 1, *p* 2, ſituated in as many different directions *OA*, *OB*, *OC* of the laſt refracted rays produced, as the glaſs has different ſurfaces *DE*, *EF*, *FG* differently inclined to the oppoſite ſurface *DH*. For theſe ſurfaces, like ſo many different priſms, give the viſual rays *PIAO*, *PKBO*, *PLCO* ſo many different bendings at *I* and *A*, *K* and *B*, *L* and *C*, and make them fall upon the eye at *O* in as many different directions *AO*, *BO*, *CO*[a]. And in all theſe inſtances when the reflecting or refracting ſurfaces of the water or glaſſes are ſhaken by the wind, or otherwiſe, the objects ſeen by reflection or refraction appear to ſhake and tremble; becauſe the laſt directions of the viſual rays are ſhaken and varied by thoſe motions.

Fig. 160 to 171.

Fig. 172.

Now the reaſon why an object or point of an object appears always in the direction of the laſt refracted or reflected ray, is, becauſe the place of its picture upon the retina is the ſame as it would be if the object was really removed from its proper place into the direction of that ray, and was ſeen by direct rays. And having no ſenſation of the previous reflections or refractions of the rays at the glaſſes, but only of their action upon a certain place of the retina, we form the ſame judgment of the place of the object as we uſed to do in the more common caſes of direct viſion. How we know and judge of the place and poſition of an object by the place and inverted poſition of its picture upon the retina, will be ſhewn in the next chapter; wherein it will appear to be entirely the effect of experience.

Apparent directions of viſible points determined. Fig. 160 to 172.

102. It is manifeſt from what has been ſaid, that any point *P* of an object *PQ* ſeen by refractions or reflections; appears ſomewhere in the line *pO*, drawn from the correſponding point *p* of its laſt image to the eye at any place *O*. Becauſe all the rays which flowed from *P* do after the laſt refraction or reflection flow from or towards the correſponding point *p* of the laſt image. The reaſon why I ſay the laſt image will be mentioned in the 111th article.

Their apparent ſituation determined.

103. It is alſo manifeſt why an object ſeen by refracted or reflected rays appears ſometimes upright and ſometimes inverted. For when the refracted or reflected rays *AO*, *CO*, have the ſame ſituation with reſpect to each other, as two rays that come directly from the ſame points of the object to the eye, theſe points will appear in the ſame ſituation with reſpect to each other in both caſes[b]. But if the rays that come from theſe points

[a] Art. 40.

[b] Art. 101.

points ſhall have croſſed each other before they arrive at the eye, they will then have a contrary ſituation to that of two rays coming directly from the ſame points to the eye; and conſequently theſe two points will appear in the glaſs in a contrary ſituation[a]. And one may add that in the former caſe, the picture upon the retina of the eye will have the ſame poſition, though not the ſame magnitude, as if the glaſs was removed, and will have a contrary poſition in the latter caſe.

a Art. 101.

Apparent magnitude in glaſſes defined.

104. The apparent magnitude of an object, PQ, ſeen by refracted or reflected rays either upright or inverted, is a quantity of viſible extenſion, meaſured by the angle, AOC, which the two [b] rays, AO, CO, that came from its extremities, P, Q, do make, after their laſt refraction or reflection, in falling on the eye. Or in other words, the object appears greater or ſmaller in proportion as that angle AOC is greater or ſmaller. Becauſe its extremities appear in the directions of the laſt refracted or reflected rays OA, OC[c]; and alſo becauſe its picture upon the retina is greater or ſmaller in proportion as theſe rays conſtitute a greater or ſmaller angle at the eye[d].

b Art. 90.

c Art. 101.

d Art. 91.

And determined.

105. Therefore the apparent magnitude of an object, PQ, is alſo meaſured by the angle pOq which its laſt image pq ſubtends at the eye. For the lines AO, pO are but one line continued, and ſo are CO, qO, and therefore the angles AOC, pOq are the ſame when the image lies before the eye, and are equal when it lies behind it.

How it varies.

106. Hence the apparent magnitude of an object increaſes and decreaſes in proportion as the eye approaches to or recedes from its laſt image, (juſt as if it was a real object[e],) placed either before or behind the eye. For when the image is fixed, the angle pOq, when ſmall, increaſes in the ſame proportion as Oq decreaſes, and on the contrary[f].

e Art. 99.

f Art. 60.

When invariable.
Fig. 173. to 176.

107. Hence if the laſt image be removed to an infinite diſtance, that is, if the object be placed in the principal focus of a lens, ſphere, or concave looking-glaſs, its apparent magnitude to the eye at any place whatever will be invariably the ſame; and equal to its apparent magnitude ſeen by the naked eye, ſuppoſing it put into the place of the center of the ſphere, lens, or reflecting concave. For ſince all the rays of any one pencil, are parallel to its axis PE, the angle COA, which meaſures the apparent magnitude at any point O, is every where equal to the angle QEP at the center E.

Fig. 177, 178.

The apparent magnitude of the object will alſo be invariable whereever it be placed, when the eye is fixed at the principal focus of any glaſs which makes parallel rays converge to the eye. For conceiving them to flow back again from the eye to the object, they will fall upon the ſame points of the object from whence they came while it is moved in any place along the axis of the glaſs: and no other rays but theſe can return from the ſame points of the object to the eye in that place;

place: therefore the ſeveral parts of the object will always be ſeen under the ſame angles, and conſequently will appear of the ſame magnitudes[a].

Compared to the true magnitude.

108. The apparent magnitude of an object ſeen by reflected or refracted rays being meaſured by the angle which its laſt image ſubtends at the eye[b], and its apparent magnitude to the naked eye in any place being meaſured by the angle which the object it ſelf ſubtends at the eye in that place[c], it follows that the former apparent magnitude is to the latter, as the former angle to the latter angle. For the meaſures of things and the things meaſured by them are proportionable.

When equal to the true.

109. Conſequently the apparent magnitude of an object ſeen in a glaſs, will be equal to its apparent magnitude to the naked eye in the ſame place, if the glaſs was removed, Firſt, when the object touches any thin lens or any ſingle ſurface. For the image is then equal to the object and coincides with it[d]. Secondly, when the eye touches any thin lens or any reflecting ſurface. For then the ray *PAO* will paſs from the object to the eye through the middle of the lens very nearly, and therefore being almoſt ſtraight[e] will make nearly the ſame angle with the axis as an unrefracted ray would do: and when the point of incidence, *A*, coincides with *C* at any reflecting ſurface, the incident and reflected rays *PA*, *AO*, produced, will alſo make equal angles with the axis or perpendicular *QC*[f]; and ſo the object will appear under the ſame angle as it would do to the naked eye turned about. Thirdly, when the eye is at the center of a reflecting concave. For then the incident and reflected rays *PA*, *AO* will coincide with the direct ray *PE*[g], and conſequently will make the ſame angles with the axis. Fourthly, when the object is at the center of the reflecting concave. For then the reflected image is alſo at the center and is equal to the object[h]. Fifthly, when a ray coming directly from *P* to *O*, would make an angle with the axis equal to the angle *AOC*, which the refracted or reflected ray *PAO* makes with it on the other ſide.

Leſs than the true through a concave and greater through a convex glaſs,

110. Theſe caſes being excepted the apparent magnitude of an object ſeen through a concave lens is always leſs than the true; and when it is ſeen upright through a convex lens, or a globe, it is greater than the true. For the ray *PAO*, coming from the extremity of the object to the eye, is bent by the concave lens from its axis, and therefore makes a leſs angle with it at the eye than a ray coming directly from that extremity to the eye. But the ſame ray is bent by the convex lens towards its axis, and therefore makes a greater angle at the eye than the direct ray: and the apparent magnitudes are meaſured by theſe angles.

The whole applyed to viſion through any number of glaſſes.

111. What has hitherto been demonſtrated concerning the apparent magnitude of an object *PQ*, will continue in force if you ſuppoſe the object *PQ* to be an image formed by another glaſs or other glaſſes. For

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[a] Art. 104.

[b] Art. 105.

[c] Art. 97.

[d] Art. 55.

[e] Art. 42.

[f] Art. 8.

[g] Art. 10.

[h] Art. 29. Fig. 165, 167, 171.

the rays diverge from either of them in the ſame manner, and for this reaſon I have always called pq the laſt image of the object.

What part of an object is viſible in any glaſs.

112. The place of the eye at O being given, to determine what part of an object is viſible in a given portion or aperture AC of any refracting or reflecting glaſs, draw OA to the edge of the aperture and produce it till it cuts the image in p, and through the center of the glaſs draw pE cutting the object in P; and PQ will be the part in view in the aperture AC. For the whole pencil of rays flowing from P will belong to p after refraction or reflection [a], and conſequently ſome one of thoſe rays will advance to the eye in the line AO drawn through p. If the image be at an infinite diſtance all the rays that belonged to p will be parallel to the axis of the pencil; therefore PQ is now determined by drawing EP parallel to OA. In a plane looking-glaſs, pP muſt be drawn from p parallel to qQ, or perpendicular [b] to the glaſs, to cut off the part PQ viſible in the aperture AC. For this glaſs may be conſidered as having a center at an infinite diſtance from it.

a Art. 43.
Fig. 173 to 176.
Fig. 168.
b Art. 25.

How it varies.

113. Hence if the glaſs and object be fixed, the part in view in a given aperture will decreaſe perpetually while the eye recedes from the glaſs; unleſs the image be behind the eye. For then it will decreaſe only till the eye arrives at the image, and after the eye has paſſed by the image it will increaſe perpetually. The reaſon is becauſe the object and image, being fixed in their places, do both increaſe or both decreaſe together, being both terminated by two lines Pp, Qq that meet or croſs in E the center of the glaſs.

When greateſt and leaſt.

114. Therefore the part in view is greateſt when the eye is cloſe to the glaſs, and leaſt when cloſe to the image; and, in this latter caſe, it appears infinitely magnified. For conceiving the diſtance Oq infinitely diminiſhed, the parts pq, PQ cut off by the lines AOp, pEP will both be infinitely diminiſhed; but the magnitude of the angle at O, ſubtended by pq or by AC, continues finite while the angle ſubtended by PQ at O is infinitely diminiſhed: and ſo the diſproportion between theſe angles, that is between the apparent and true magnitudes of the particle PQ [c] is infinitely great. It appears alſo infinitely confuſed, when the pupil is open, for the reaſon given in the following articles.

c Art. 108.

The ſize of a looking-glaſs ſufficient to ſee all ones own body.

115. When a perſon views himſelf in a plane looking-glaſs he appears to himſelf to fill the ſame part of the glaſs wherever he ſtands: and the length and breadth of this part is always half the length and breadth of the correſponding part of his own body. For when O and Q coincide, OC is half of Oq or Qq [d] and therefore AC is half of pq [e] or PQ.

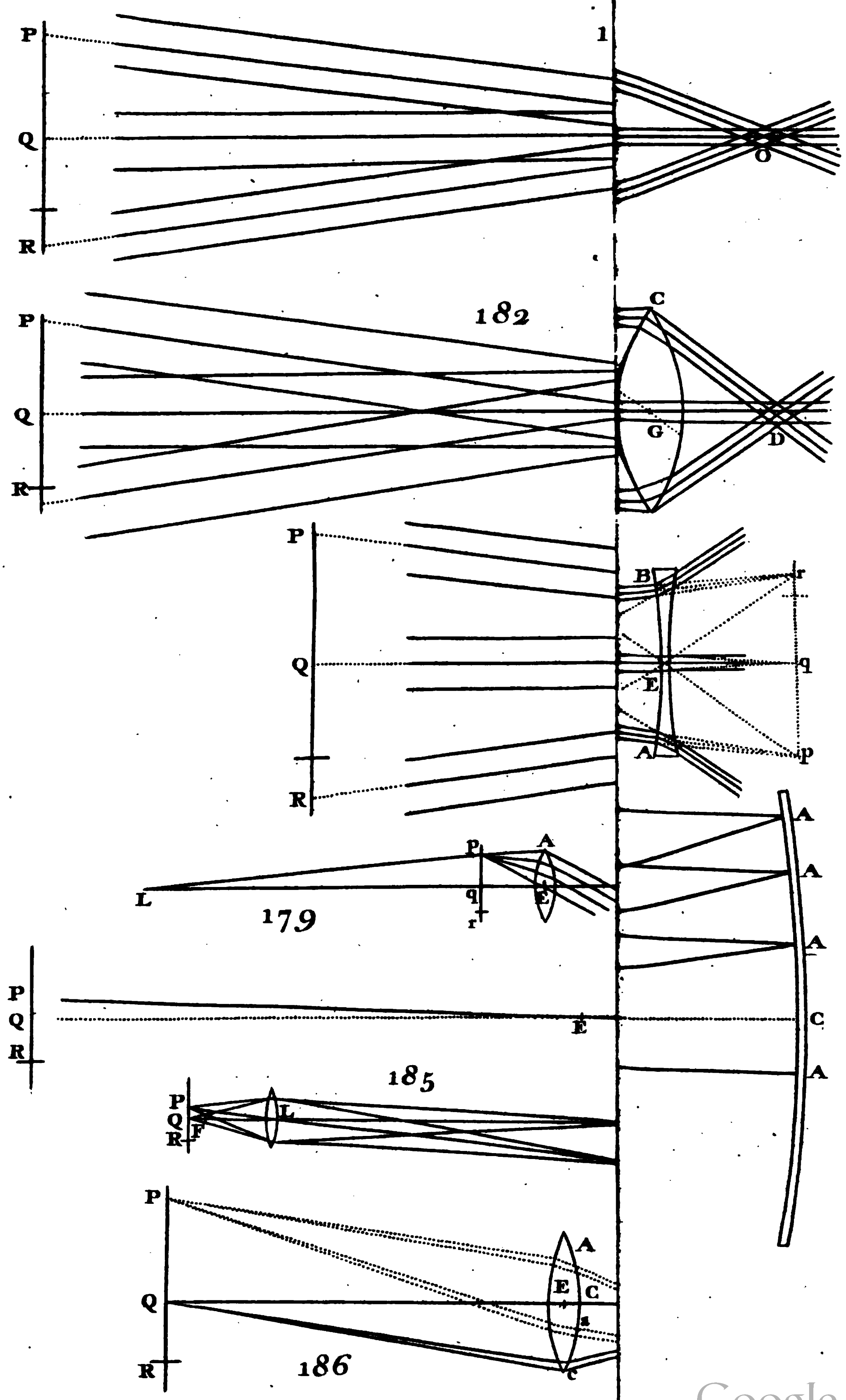
Fig. 168.
d Art. 23.
e Art. 57.

Viſion when confuſed by glaſſes.

116. Hitherto I have conſidered the pupil of the eye as no bigger than a point, admitting but a ſingle ray from every point of the object [f]; by which means the picture upon the retina would be diſtinct in all caſes.

f Art. 90.

But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But when the pupil is open, if the image formed by the glaſs be nearer to the eye than the leaſt diſtance at which we can ſee objects diſtinctly with the naked eye, the appearance through the glaſs will be confuſed. Becauſe the rays diverge too much from ſo near an image to be reduced by the eye to a diſtinct picture upon the retina. On the other hand, when the rays converge to an image behind the eye, they will be collected to a diſtinct picture before they arrive at the retina, becauſe the eye is not naturally uſed to conform it ſelf to converging rays; and ſo the viſion will be confuſed in both caſes, but may be rendered diſtinct as followeth.

How rendered diſtinct.

117. Things which appear confuſed when ſeen by direct, refracted or reflected rays may be rendered diſtinct either by looking through a little hole in a thin plate or bit of paper, or through a convex or concave glaſs of a proper degree of convexity or concavity; and provided the hole or glaſs be put cloſe to the eye, the apparent magnitude and ſituation of the object will be the ſame in both caſes. For if the hole be ſo ſmall as to admit but a ſingle ray from every diſtinct point of the object, theſe rays will fall upon the retina in as many other diſtinct points, and will make a diſtinct picture. And when the pencils of rays fall upon a thin lens, their axes go ſtraight through the middle of it[a], and conſequently will proceed to the ſame points upon the retina as when they paſſed through the hole. Now ſuppoſing the lens to have ſuch a figure that the rays of every pencil ſhall be refracted by it, and by the eye together, to thoſe very points of their axes, which touch the retina, the picture will ſtill be diſtinct: and will be the ſame in magnitude and poſition as before: and the only difference in the effects of the hole and lens will be in the degree of brightneſs of the picture upon the retina.

[a] Art. 43.

A ſingle microſcope, how much it magnifies.

Fig. 179, 180.

118. A ſingle microſcope is only a very ſmall globule of glaſs, or a ſmall double convex glaſs, whoſe focal diſtance is very ſhort. A minute object pq ſeen diſtinctly through a ſmall glaſs AE by the eye put cloſe to it, appears ſo much greater than it would to the naked eye, placed at the leaſt diſtance qL from whence it appears ſufficiently diſtinct, as this latter diſtance qL is greater than the former qE. For having put your eye cloſe to the glaſs EA, in order to ſee as much of the object as poſſible at one view[b], remove the object pq to and fro till it appears moſt diſtinctly, ſuppoſe at the diſtance Eq. Then conceiving the glaſs AE to be removed, and a thin plate, with a pin-hole in it, to be put in its place, the object will appear diſtinct, and as large as before[c], when ſeen through the glaſs, only not ſo bright. And in this latter caſe, it appears ſo much greater than it does to the naked eye, at the diſtance qL, either with the pin-hole or without it, as the angle pEq is greater than the angle pLq[d], or as the latter diſtance qL is greater than the former qE[e].

[b] Art. 114.

[c] Art. 117.

[d] Art. 97.

[e] Art. 60.

119. Since

And in what manner.

119. Since the interpoſition of the glaſs has no other effect than to render the appearance diſtinct, by helping the eye to increaſe the refraction of the rays in each pencil, it is plain that the greater apparent magnitude is intirely owing to a nearer view than could be taken by the naked eye. If the eye be ſo perfect as to ſee diſtinctly by pencils of parallel rays falling upon it, the diſtance Eq, of the object from the glaſs, is then the focal diſtance of the glaſs. Now if the glaſs be a ſmall round globule whoſe diameter is $\frac{1}{15}$ of an inch, ſuch as are eaſily made in the manner deſcribed in the third book, its focal diſtance Eq being three quarters of its diameter, [a] is $\frac{1}{20}$ of an inch; and if qL be 8 inches, the uſual diſtance at which we view minute objects, this globule will magnify at the rate of 8 to $\frac{1}{20}$ or of 160 to 1.

Aſtronomical teleſcope, how much it magnifies, and why. Fig. 181.

120. An aſtronomical teleſcope is compoſed of two convex glaſſes in the following manner. PQ repreſents the ſemidiameter of a remote object, and pq its picture formed by the convex lens L, which being next to the object is called the object-glaſs. In the axis of this glaſs, QLq produced, EA repreſents another glaſs more convex than L, ſo placed, that as qL is the focal diſtance of the glaſs L, ſo qE is the focal diſtance of the glaſs E; and EL the ſum of their focal diſtances. In this ſituation of the glaſſes, I ſay the object will appear to the eye at any point O, diſtinct, inverted and magnified at the rate of qL to qE, that is of the focal diſtance of the object glaſs to the focal diſtance of the eye glaſs.

For the rays which diverge from the point q of the picture pq, being refracted by the eye-glaſs, will emerge upon the eye at O in lines parallel to the axis qEO; becauſe qE is ſuppoſed to be the focal diſtance of the eye-glaſs; and for the ſame reaſon, the rays which diverge from any collateral point p, of that picture pq, will emerge from the eye-glaſs, after refractions at A, in lines parallel to the line or ray pE; this line being the axis of an oblique pencil of rays, part of which diverge from p upon the glaſs. An eye therefore which can ſee diſtinctly by pencils of parallel rays being placed any where at O, among the interſections of theſe pencils, will ſee the points of the object diſtinctly.

Now to the eye at O the apparent magnitude of the picture pq, or object PQ, is meaſured by the angle EOA[b], or by the equal angle qEp; but to the naked eye at L, if the glaſs was removed, the apparent magnitude of the object is meaſured by the angle QLP, or by the equal angle qLp; the oblique axis PLp being ſtraight[c]. Therefore the former apparent magnitude is to the latter, as the angle qEp, to the angle qLp; and conſequently as the latter diſtance qL, to the former qE[d].

A teleſcope made of four convex glaſſes conſidered. Fig. 182.

121. The object which appeared inverted in the former teleſcope[e], will appear upright and diſtinct through two more convex eye-glaſſes ſubjoined to it; at a diſtance from each other, equal to the ſum of their focal

[a] Art. 61.

[b] Art. 104.

[c] Art. 43.

[d] Art. 60.

[e] Art. 103.

focal diſtances; and when their focal diſtances are equal to each other, the object will be magnified juſt as much as it was before. For the pencils of parallel rays EOF, AOB, &c, which are continued to the glaſs FB, will be formed by it into a ſecond image $\varpi\kappa$; and the focus ϖ, of any oblique pencil OB, will be determined by the interſection of the line $\varpi\kappa$, perpendicular to the common axis of the glaſſes, and of the oblique axis $F\varpi$, drawn parallel to the incident rays OB[a]. This point ϖ being the focus of incident rays on the laſt glaſs GC, the emergent rays CD will be parallel to their oblique axis ϖG; becauſe the rays that flow from κ are ſuppoſed to emerge parallel to the direct axis. Therefore to the eye at D, where theſe emergent pencils croſs, the object will appear diſtinct, and upright[b]. And when the glaſſes F and G are exactly equal, the image $\varpi\kappa$ will be exactly in the middle between them; and ſo the triangles $\varpi F\kappa$, $\varpi G\kappa$ will be exactly equal. Conſequently the angle CDG, which now meaſures the apparent magnitude to the eye at D, will be equal to the angle $\varpi G\kappa$ or $\varpi F\kappa$ or BOF or AOE, which meaſured it before to the eye at O.

How much they take in at one view.
Fig. 181, 182.

122. In a teleſcope of a given length the quantity of objects taken in at one view, depends upon the breadth of the eye-glaſs. For as AE is greater or ſmaller, the angle ALE or its equal PLQ is alſo greater or ſmaller; and this angle takes in all the objects that can be ſeen at one view on one ſide of the axis of the teleſcope.

Galileo's teleſcope conſidered.
Fig. 183.

123. The difference between the aſtronomical teleſcope and *Galileo*'s teleſcope or a common perſpective-glaſs is this; inſtead of the convex eye-glaſs placed behind the image to make the rays of each pencil go parallel to the eye, there is placed a concave eye-glaſs AE as much before it; which opens the rays of each pencil that converged to q and p, and makes them emerge parallel upon the eye; as is evident by conceiving the rays to go back again through the eye-glaſs, whoſe focal diſtance we ſuppoſed was Eq. The eye muſt be put cloſe to the glaſs to receive as many pencils as poſſible; and then, ſuppoſing an emerging ray of an oblique pencil produced backward along AO, the apparent magnitude of the object is meaſured by the angle AOE[c] or its equal qEp, which is to the angle qLp (or QLP, the meaſure of the true magnitude,) as qL to qE, as before in the other teleſcope. It is manifeſt, by the 103d article, that objects in this teleſcope appear upright.

This takes in leſs than the former.

124. The quantity of objects taken in at one view in this teleſcope does not depend upon the breadth of the eye-glaſs, as in the aſtronomical teleſcope, but upon the breadth of the pupil of the eye. Becauſe the pupil is leſs than the eye-glaſs, and the lateral pencils do not now converge to but diverge from the axis of the glaſſes. Upon this account the view being narrower is not ſo pleaſant as in the former teleſcope.

125. Sir

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[a] Art. 55.

[b] Art. 103.

[c] Art. 104.

Sir *Iſ. Newton*'s reflecting teleſcope. Fig. 184.

125. Sir *Iſaac Newton*'s reflecting teleſcope magnifies the diameter of a remote object in the proportion of the focal diſtance of the reflecting concave to the focal diſtance of the convex eye-glaſs and ſhews it inverted. Let ST be an image of a remote object PQ formed by reflections from a large concave ſurface AC, and terminated by the lines $PESA$, $QETC$ drawn through its center E. Now becauſe this image cannot be viewed through an eye-glaſs placed directly before it (for then the ſpectator would intercept the rays that are coming to the concave) therefore let the ſeveral pencils of rays which converge towards it in coming from the broad concave AC, be reflected ſideways from a ſmall poliſhed plane, repreſented by ac; and then the ſecond image st, formed by this plane, will be equal to the firſt image ST[a]. Let tl be the focal diſtance of a ſmall convex eye-glaſs kl and the rays which flow from any point s will be refracted through this glaſs, to the eye at o, in the lines ko drawn parallel to the oblique axis sl; and ſo the apparent magnitude of the object, PQ, to the eye at o, will be meaſured by the angle kol or slt[b]: but to the naked eye at E, it is meaſured by the angle PEQ or SET. Therefore the former apparent magnitude is to the latter, as the angle slt to the angle SET or, (becauſe their ſubtenſes st, ST are equal,) as ET to lt[c] or as CT to lt, when the object is remote[d]. Note that the plane acb is much too broad in compariſon to the concave ACB, which could not be helped in ſo ſmall a draught. That the appearance of the object is inverted or turned from right to left, is evident by the 103d article.

[a] Art. 24, 25.
[b] Art. 104.
[c] Art. 60.
[d] Art. 26.

Why ſo much ſhorter than others.

126. Dioptrick teleſcopes which magnify much being very long and troubleſome to be managed, Sir *Iſaac Newton* propoſed this method to ſhorten teleſcopes[1]: which anſwers to admiration; as appears by a table in the next book, of the lengths of both ſorts of teleſcopes which magnify equally with equal diſtinctneſs. The reaſon why dioptrick teleſcopes cannot be ſhortened as much as theſe, and ſtill magnify as much, by diminiſhing the focal diſtances of the eye-glaſſes[e], in ſhort is this. The images made by refractions through the convex object-glaſſes, being much more imperfect than thoſe, which are made by reflections from concave ſurfaces, will not bear to be magnified ſo much by ſo ſmall eye-glaſſes[f], without appearing confuſed: and the chief cauſe of thoſe imperfections in the pictures is the unequal refrangibility of rays of different colours; as will be fully explained hereafter.

[e] Art 120.
[f] Art. 118.

Double microſcope conſidered. Fig. 185.

127. A double microſcope is compoſed of two convex glaſſes placed at E and L. The glaſs L next the object PQ is very ſmall and very much convex, and conſequently its focal diſtance LF is very ſhort; the diſtance LQ of the ſmall object PQ is but a little greater than LF; ſo that the image pq may be formed at a great diſtance from the glaſs[g],

[g] Art. 48.

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[1] Opticks. p. 95.

and conſequently may be much greater than the object it ſelf[a]. This picture pq being viewed through a convex eye-glaſs AE, whoſe focal diſtance is qE, appears diſtinct as in a teleſcope. Now the object appears magnified upon two accounts; firſt becauſe if we viewed its picture pq, with the naked eye, it would appear as much greater than the object, at the ſame diſtance, as it really is greater than the object, or as much as Lq is greater than LQ[b]; and ſecondly becauſe this picture appears magnified through the eye glaſs as much as the leaſt diſtance at which it can be ſeen diſtinctly with the naked eye, is greater than qE, the focal diſtance of the eye-glaſs[c]. For example, if this latter ratio be 5 to 1, and the former ratio of Lq to LQ be 20 to 1, then upon both accounts the object will appear 5 times 20, or 100 times greater than to the naked eye.

[a] Art. 55.

[b] Art. 55.

[c] Art. 118.

To fit teleſcopes and microſcopes to defective eyes.

128. To fit theſe teleſcopes and microſcopes to ſhort-ſighted eyes, the glaſſes E and L muſt be placed a little nearer together; ſo that the rays of each pencil may not emerge parallel but may fall diverging upon the eye[d]; and then the apparent magnitude will be altered a little but ſcarce ſenſibly.

[d] Art. 48.

The apparent brightneſs through them.

129. The brightneſs of the appearance through a given teleſcope or microſcope is more or leſs in proportion to the aperture of the object-glaſs. For ſuppoſing it covered with paper, all but a ſmall hole in the middle, the magnitudes of the pictures pq in the focus of the glaſſes, and of that upon the retina would not be altered; but the hole at L being ſmaller than before, there are fewer rays in every pencil, and conſequently in every point of thoſe pictures, and ſo they appear more obſcure. If the aperture and object-glaſs remain the ſame, things appear brighter or fainter according as the focal diſtance of the eye-glaſs is longer or ſhorter; that is, according as the teleſcope or microſcope magnifies leſs or more[e]. For the ſame quantity of light ſpread over a ſmaller or larger picture or part of the retina will make it brighter or duller.

[e] Art. 120, 127.

All appearances the ſame when the eye is out of the axis of the glaſs or glaſſes. Fig. 186.

130. Hitherto I have ſuppoſed the eye to be always placed at ſome point O in the common axis of the refracting or reflecting ſurfaces. Now let it be placed at any point o in a line Oo perpendicular to the axis Qq; I ſay that all the appearances will be the ſame or at leaſt not ſenſibly different from what they were before. For let pq be the laſt image of an object, and PQ the laſt but one, or the object it ſelf; draw the lines po, qo meeting the next ſurface in a and c; and the points P and Q will appear to the eye at o in the directions of thoſe lines oa, oc. Whence drawing pO meeting the ſurface in A; ſince the directions OA, oa, in which P is ſeen, lye the ſame way from the directions OC, oc, in which Q is ſeen, it is evident that the apparent ſituation of the extremities P, Q is the ſame at both places of the eye; and alſo the apparent magnitude, which is meaſured by the angle aoc[f] or poq or pOq or AOC. For the

[f] Art. 124.

ſmall angles poq, pOq, being ſubtended by the ſame image pq, very nearly at equal diſtances po, qO from o and O, are very nearly equal. The apparent brightneſs of the object is alſo the ſame; becauſe the denſity of the rays, that enter the pupil, at any part of the perpendicular plane repreſented by Oo, is nearly the ſame[a]. For the rays flow from or towards the laſt image pq juſt as if it was a luminous body. And laſtly the degree of apparent diſtinctneſs or confuſion is the ſame alſo, becauſe the angles which the pupil, placed at O or at o, ſubtends at p and q, or the mutual inclinations of the rays in each pencil are very nearly equal.

[a] Art. 58.

A general obſervation upon viſion.

131. This general obſervation upon viſion is worth remembring. That the apparent diſtinctneſs and confuſion of an object depends upon the mutual inclination of the rays to each other in any one pencil when they fall upon the eye[b]; the apparent magnitude, upon the inclination of the rays of different pencils to each other when they fall upon the eye[c]; the apparent ſituation, upon the real ſituation of the extream pencils when they fall upon the eye[d]; and the apparent brightneſs and obſcurity, upon the quantity of rays in every pencil[e].

[b] Art. 116.
[c] Art. 104.
[d] Art. 103.
[e] Art. 68.

CHAP. V.

CONCERNING OUR IDEAS ACQUIRED BY SIGHT.

The caſe of perſons born blind conſidered.

132. IN order to account for ſeveral appearances in viſion, it is neceſſary to conſider the manner of acquiring our ideas of things by ſight. The noted queſtion propoſed by Mr. *Molyneux* to Mr. *Locke*, whether a perſon blind from his birth, being made to ſee, could by ſight alone diſtinguiſh a globe from a cube, whoſe difference he knew by feeling, has been pronounced in the negative by both thoſe philoſophers[1]: and this opinion has ſince been confirmed by the experience of ſeveral perſons, who receiving their ſight from the operation of Couching, could not know any one thing from another, however different in ſhape and magnitude. Mr. *Cheſſelden* having given us a very curious account of ſome obſervations made by a young gentleman who was couched by him in the thirteenth year of his age, I will here inſert it in his own words[2].

Mr. *Cheſſelden*'s account of ſuch perſons brought to ſight.

133. Though we ſay of this gentleman that he was blind, as we do of all people who have ripe Cataracts, yet they are never ſo blind from that cauſe, but that they can diſcern day from night; and for the moſt part in a ſtrong light, diſtinguiſh black, white, and ſcarlet, but they cannot perceive the ſhape of any thing: for the light by which theſe perceptions are made, being let in obliquely through the aqueous humour,

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[1] *Locke*'s Eſſay on Hum. Underſt. B. 1. c. 9.

[2] Phil. Tranſ. No. 402. See another account in the 55th Tatler.

or the anterior ſurface of the cryſtalline (by which the rays cannot be brought into a focus upon the retina) they can diſcern in no other manner, than a ſound eye can through a glaſs of broken jelly where a great variety of ſurfaces ſo differently refract the light, that the ſeveral diſtinct pencils of rays cannot be collected by the eye into their proper foci; wherefore the ſhape of an object in ſuch a caſe, cannot be at all diſcerned, though the colour may: and thus it was with this young gentleman, who though he knew theſe colours aſunder in a good light; yet when he ſaw them after he was couched, the faint ideas he had of them before, were not ſufficient for him to know them by afterwards; and therefore he did not think them the ſame, which he had before known by thoſe names. Now ſcarlet he thought the moſt beautiful of all colours, and of others the moſt gay were the moſt pleaſing; whereas the firſt time he ſaw black, it gave him great uneaſineſs, yet after a little time he was reconciled to it: but ſome months after, ſeeing by accident a Negro woman, he was ſtruck with great horror at the ſight.

When he firſt ſaw, he was ſo far from making any judgment about diſtances that he thought all objects whatever touched his eyes (as he expreſſed it) as what he felt did his ſkin; and thought no objects ſo agreeable as thoſe which were ſmooth and regular, though he could form no judgment of their ſhape, or gueſs what it was in any object that was pleaſing to him. He knew not the ſhape of any thing, nor any one thing from another, however different in ſhape, or magnitude, but upon being told what things were, whoſe form he before knew from feeling, he would carefully obſerve, that he might know them again; but having too many objects to learn at once, he forgot many of them: and (as he ſaid) at firſt he learned to know, and again forgot a thouſand things in a day. One particular only (though it may appear trifling) I will relate: Having often forgot which was the cat, and which the dog, he was aſhamed to aſk, but catching the cat (which he knew by feeling) he was obſerved to look at her ſtedfaſtly, and then ſetting her down, ſaid, ſo puſs, I ſhall know you another time. He was very much ſurprized, that thoſe things which he had liked beſt, did not appear moſt agreeable to his eyes, expecting thoſe perſons would appear moſt beautiful that he loved moſt, and ſuch things to be moſt agreeable to his ſight that were ſo to his taſte. We thought he ſoon knew what pictures repreſented, which were ſhewed to him, but we found afterwards we were miſtaken: for about two months after he was couched he diſcovered at once, they repreſented ſolid bodies; when to that time he conſidered them only as party-coloured planes, or ſurfaces diverſified with variety of paint; but even then he was no leſs ſurprized, expecting the pictures would feel like the things they repreſented, and was amazed when he found thoſe parts, which by their light and ſhadow appeared now round and uneven, felt only flat like the reſt: and aſked which was the lying ſenſe, feeling or ſeeing?

Being ſhewn his father's picture in a locket at his mother's watch, and told what it was, he acknowledged a likeneſs, but was vaſtly ſurprized; aſking how it could be, that a large face could be expreſſed in ſo little room; ſaying, it ſhould have ſeemed as impoſſible to him, as to put a buſhel of any thing into a pint.

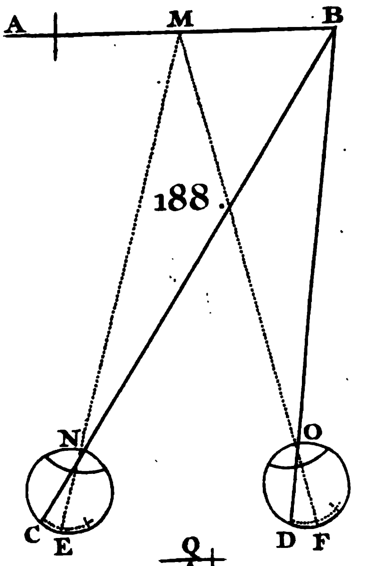
At firſt he could bear but very little light, and the things he ſaw, he thought extreamly large; but upon ſeeing things larger, thoſe firſt ſeen he conceived leſs, never being able to imagine any lines beyond the bounds he ſaw: the room he was in, he ſaid, he knew to be but part of the houſe, yet he could not conceive that the whole houſe could look bigger. Before he was couched, he expected little advantage from ſeeing, worth undergoing an operation for, except reading and writing; for he ſaid, he thought he could have no more pleaſure in walking abroad than he had in the garden, which he could do ſafely and readily. And even blindneſs he obſerved, had this advantage, that he could go any where in the dark much better than thoſe who can ſee; and after he had ſeen, he did not ſoon loſe this quality, nor deſire a light to go about the houſe in the night. He ſaid every new object was a new delight, and the pleaſure was ſo great, that he wanted ways to expreſs it; but his gratitude to his operator he could not conceal, never ſeeing him for ſome time without tears of joy in his eyes, and other marks of affection: and if he did not happen to come at any time when he was expected, he would be ſo grieved, that he could not forbear crying at the diſappointment. A year after his firſt ſeeing, being carried upon *Epſom* Downs, and obſerving a large proſpect, he was exceedingly delighted with it, and called it a new kind of ſeeing. And now being lately couched of his other eye, he ſays, that objects at firſt appeared large to this eye, but not ſo large as they did at firſt to the other: and looking upon the ſame object with both eyes, he thought it looked about twice as large as with the firſt couched eye only, but not double, that we can any ways diſcover.

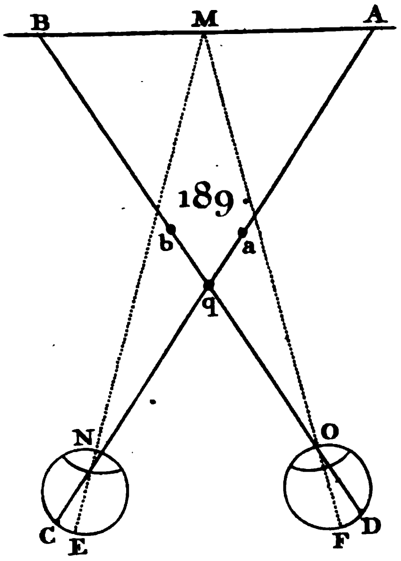
Some additions to that account.

134. Mr. *Cheſſelden* adds in another paper printed by it ſelf, that he has brought to ſight ſeveral others who had no remembrance of ever having ſeen; and that they all gave the ſame account of their learning to ſee, as they called it, with the young gentleman above mentioned, though not in ſo many particulars; and that they all had this in common, that having never had occaſion to move their eyes they knew not how to do it; and at firſt, could not at all direct them to a particular object; but in time they acquired that faculty, though by ſlow degrees.

By what regular ſteps they may learn to know things.

135. Let us now conſider a little by what regular ſteps and obſervations a perſon in this caſe, might learn to know the places, magnitudes, figures,


188.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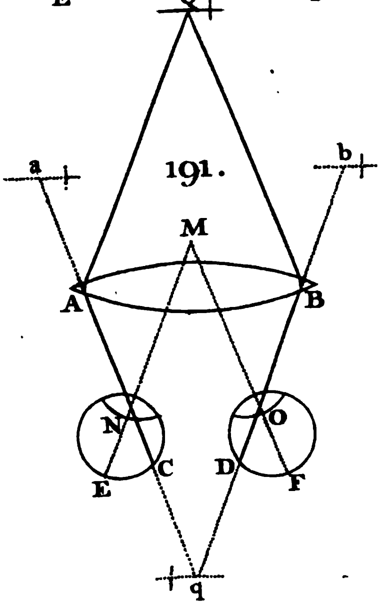
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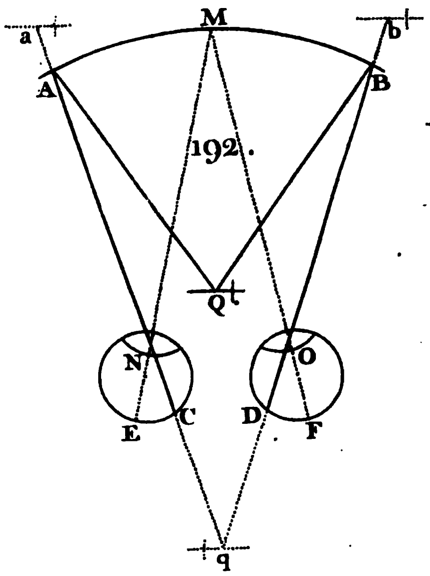
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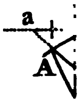

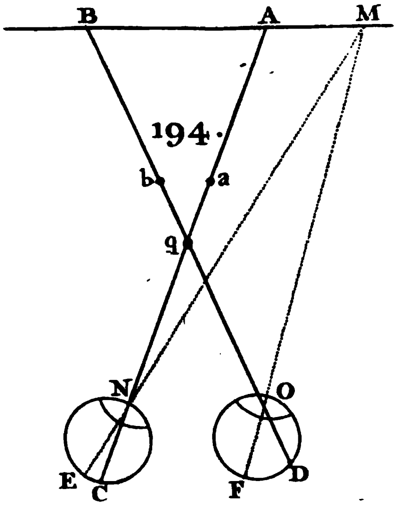
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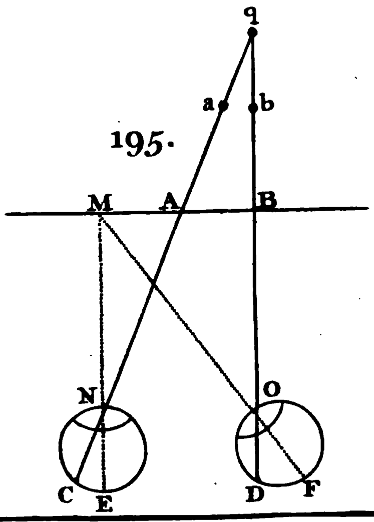
195.

figures and diſtances of objects. Since he cannot direct his eye to look at any particular object[a], whoſe place he knows by feeling, at firſt we muſt ſuppoſe his eye at reſt; and when he has learned to know his hand, or his finger end, let him move it gently upwards and downwards. During this motion he cannot help perceiving ſome ſort of alteration in the viſible appearance, occaſioned by the correſponding motion of the picture of his finger over different parts of the retina. Then by carefully obſerving and remembring what ſort of ſenſation was perceived when his finger was in any particular place, ſuppoſe above his eye; whenever the like ſenſation ſhall again be excited, by another picture of the ſame or of a different object, falling upon the ſame place of the retina, wherever it be, he will conclude that this object, whoſe place is unknown, is above his eye, or in the place where he formerly held his finger. By the like obſervations made with his hand and frequently repeated, he may learn to know the motion of a body by ſight; and the direction of its motion, with reſpect to his own body; and conſequently to know extenſion and the ſituation of extenſion; and conſequently to know the figure of bodies, which conſiſts only of various extenſions variouſly ſituated. This he may learn by moving his finger round the edges of bodies, and by obſerving the various bendings of its viſible motion; or by walking round a room; and in general by comparing together the ideas ſuggeſted to his mind by ſight and by touch. And by obſerving that the appearance of the ſame body is continually varied while the eye approaches to it or recedes from it, by this variety of apparent magnitude he will learn to know the diſtances of things from his eye as well as from one another. Laſtly ſince he cannot help perceiving thoſe objects moſt diſtinctly which lye neareſt to the axis of the eye produced, and others more confuſedly which lye farther and farther from it, as other people do; when he finds that an object diſtinctly perceived, does ſuddenly grow confuſed, by any accidental motion of his head or eye, the memory of that diſtincter perception now loſt will induce him to try to regain it, by a voluntary motion of his head or eye; till by frequent trials he will learn to direct his eye to any deſired object. And by the ſame method he will learn to direct both eyes to the ſame object. From what has been ſaid it appears that our perception of things by ſight is no more than this: by memory of former perceptions by ſight and other ſenſes compared together, we collect in an inſtant that the thing we now perceive by ſight only will affect our other ſenſes, upon trial, as it formerly uſed to do. I ſay in an inſtant, which will leſs ſurprize us, when we conſider how quick the characters or ſounds of words, whoſe ſignification we could hardly remember at firſt, do excite in our minds the ideas of things they are conſtantly uſed to ſignify: ſo great is the force of habits in bringing our ideas together. And ſo it appears at

[a] Art. 134.

at laſt, that the manner, wherein external objects are ſignified to us, by the ſenſations of light and colours, is the ſame with that of languages and ſigns of human appointment: which do not ſuggeſt the things ſignified by any likeneſs or identity of nature, but only by an habitual connection that conſtant experience has made us obſerve between them *.

The effect of inverted pictures on the retina conſidered.

136. Now if it be the memory of the ſame ſenſations excited in the ſame places of the retina, though unknown, which occaſions the ſame judgment of the place of an object, (which will be farther confirmed in the following articles,) the inverted pictures on the retina will ſerve as well to excite the ſame ideas as if they had been conſtantly upright or in any oblique poſture. It is only neceſſary that the object and picture ſhould alter their places both together by any conſtant rule whatever. For example, if a perſon was born with an eye in which the pictures of objects were painted upright, as repreſented in the 187th figure, where the firſt image *pq* is formed by the cornea in the aqueous humour enlarged, and the ſecond image by the cryſtalline upon the retina ϖx; let it be conſidered whether by theſe upright pictures he might not learn to know, judge, and talk of objects juſt as other people do, and on the contrary; and whether it could eaſily be known that his eyes were different from common eyes.

Viſion with both eyes when ſingle when double. Fig. 188.

137. The axis of the eye is a line drawn through the middle of the pupil and of the cryſtalline, and conſequently falls upon the middle of the retina. And the axes of both eyes produced are called the optick axes. When the optick axes are parallel or meet in a point, the two middle points of the retinas, or any two points which are equally diſtant from them, and lye on the ſame ſides of them either towards the right hand or left hand, or upwards or downwards, or in any oblique direction, are called correſponding points. Now we find by experience that an object or point of an object appears ſingle when its pictures fall upon correſponding points of the retinas, and double when they do not. For when we view an object ſteadily, we have acquired a habit of directing the optick axes to the point in view; becauſe its pictures falling upon the middle points of the retinas are then diſtincter than if they fell upon any other places; and ſince the pictures of the whole object are equal to one another and are both inverted with reſpect to the optick axes, it follows that the pictures of any collateral point of the object are painted upon correſponding points of the retinas. This habit of directing the optick axes to the point in view, is ſo ſtrong that it is very difficult to do otherwiſe; inſomuch that when one eye is ſhut and the other is in motion, one may feel by ones fingers laid upon the eye-lid, that the eye which is ſhut, always follows the motions of the eye that is open. But if by ſquinting or by depreſſing an eye with ones finger, the

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* See *Berkeley*'s Eſſay on Viſion. p. 172.

optick axes are not directed to the ſame point; in theſe caſes objects appear double: and now it is plain that their pictures are not painted upon correſponding places of the retinas.

For the like reaſon if while the optick axes *NM*, *OM* are directed to a mark *M*, we attend to an object or image *q*, placed any where within the angle *NMO* or its oppoſite, made by the optick axes produced, the object *q* will appear in two places, ſuppoſe at *a* and *b*, ſituated in the directions of the viſual rays *Nq*, *Oq*[a]. For the pictures of the object, *q*, which lyes between the optick axes, being both inverted with reſpect to the axes, muſt fall upon the retinas on contrary ſides of the axes, and conſequently upon places that do not correſpond. And this may be the reaſon of the double appearance. Becauſe this ſituation of the pictures never happens in the ordinary and conſtant uſe of our eyes, but from two objects *A* and *B* placed on oppoſite ſides of the mark *M*. One of theſe objects alone being on the ſame ſides of both axes, would have its pictures on correſponding points of the retinas and conſequently would appear ſingle. Add to this, that at one view we generally regard no other objects but thoſe which lye round about the mark *M*, much at equal diſtances from the eyes; and not thoſe which lye in a long line extended from us. For theſe objects being placed at various diſtances from the eye cannot be ſeen diſtinctly all at once: it being neceſſary for this purpoſe to alter both the diſtances of the point of concourſe of the optick axes, and alſo the configuration of the eye; that the pictures formed by rays coming from different diſtances, may be ſucceſſively diſtinct. Fig. 189, 190.

In like manner if the image *q* of an object *Q* be formed at any place behind the eyes, either by reflected or refracted rays; and the glaſs *AB* be ſufficiently broad to throw the rays into both eyes; the object *Q* will always appear double. For to procure diſtinct viſion we are accuſtomed to direct the optick axes to ſome point *M* lying before them. But the viſual rays, *ANq*, *BOq*, by which the object is ſeen, are tending to meet at *q* behind the eyes, and conſequently muſt fall at *C* and *D* on the inſides of their axes, which are not correſponding places. Fig. 191, 192, 193.

I find by experience that the apparent diſtance between the two apparent places of the object, is nearly proportionable to the ſum of the arches, *CE*, *DF*, upon the retinas; or to the ſum of the angles *aNM*, *bOM* made by each optick axis and each viſual ray, provided theſe arches lye both on the inſides or both on the outſides of the optick axes: but if one lyes within and the other without, the apparent diſtance of the places of the object is meaſured by the differences of thoſe arches. For though I have hitherto ſuppoſed the object to be within the angles made by the optick axes, in order to ſhew the effect of the double appearance more plainly, yet it is ſtill double though the object be placed in either axis, or on the outſides of both either much nearer or remoter than Fig. 194, 195.

their

[a] Art. 101.

their point of concourſe. I find alſo that in all ſituations, the apparent interval, between the two apparent places, will continue the ſame while the eyes are rolled about, ſo as to view any objects placed nearly at equal diſtances from them: and that each image *a* or *b* appears over againſt the ſame object *A* or *B* when both eyes are open, as it does when the Fig. 189. other eye is ſhut. And that when the object, or image *q* formed by the glaſs, is between the eyes and the mark we look at, the apparent image on the right hand is perceived by the left eye, and the apparent image on the left hand by the right eye; as is manifeſt by opening and ſhut- Fig. 190. ting each eye by turns: but when the object it ſelf, or its image formed by the glaſs, is beyond the mark, or behind the eyes, the apparent image on the right hand is perceived by the right eye, and that on the left hand by the left eye.

Hence it is manifeſt that the two apparent places *a*, *b* of the object *q* are neither of them the ſame as its real place; and that they lye between it and the mark we look at, but not very far from the real place.

Fig. 196. A double appearance will alſo be ſeen when the end of a long ruler is placed between the eye brows and extended directly forward with its flat ſides reſpecting the right hand and the left; and by directing the eye to a remote object, the right ſide of the ruler, ſeen by the right eye, will appear on the left hand, and the left ſide on the right hand; as repreſented in the 196th figure; in which *PQ* denotes the ruler, *pq* and ϖx its images ſeen by the eyes *N*, *O* reſpectively.

Now if it be aſked why in ſeeing with both eyes we do not always ſee double, becauſe of a double ſenſation; I think it is ſufficient to ſay that in the ordinary uſe of our eyes, in which the pictures of an object are conſtantly painted upon correſponding places of the retinas, the predominant ſenſe of feeling has originally and conſtantly informed us that the object is ſingle. By this means our idea of its outward place is connected with both thoſe ſenſations, as is manifeſt by its appearing in two places when its pictures are not painted upon correſponding places of the retinas in the extraordinary circumſtances above mentioned; which is only a direct conſequence ariſing from our general habit of ſeeing. Beſides, whatever anſwer is ſufficient to this queſtion, muſt equally ſerve by the rules of philoſophy, for an anſwer to all others of the ſame ſort: as how it happens that in hearing with two ears we do not hear double; that in feeling with two feet or two hands or two fingers, we do not feel double; as we really do in the dark, when a button is preſſed with two oppoſite ſides of two contiguous fingers laid acroſs; for this reaſon, that thoſe oppoſite ſides of the fingers have never been uſed to feel one but always two things at a time. We have learned therefore by experience of both ſenſes compared together, to make their informations conſiſtent with each other. Mr. *Cheſſelden* mentions the caſe of a gentleman, who having had one of his eyes diſtorted by a blow on his head, found that every object appeared double; but by degrees the moſt familiar ones became ſingle, and in time all objects became ſo, though the diſtortion ſtill continued *: which greatly confirms the preſent argument, that the judgments we make of the number and places of external objects are entirely the effect of experience; by which our ideas of their number and places are conſtantly connected with certain ſenſations in correſponding places of the retinas: inſomuch that if an animal had a hundred eyes, in the ordinary and conſtant uſe of them an object would appear to him ſingle, and centuple in caſes extraordinary, like thoſe above mentioned.

Apparent diſtance what and how perceived.

138. The apparent diſtance of an object, perceived by ſight, is an idea of a real diſtance uſually meaſured by feeling, as by the motion of the body in walking, or otherwiſe; and is ſuggeſted to the mind by the apparent magnitude of the object in view, if ſeen alone, (as a bird in the air, or as an object in a teleſcope or microſcope;) but if it be ſeen with other objects, as it uſually happens, its diſtance is ſuggeſted both by its own apparent magnitude and by the apparent magnitudes of other adjoining objects obliquely extended between the eye and the object in view; as the ſurface of the ground, rivers, walks, high-ways, hedges and ditches, or the houſes in a ſtreet, or the walls and ceiling of a room, or the ſky over head. For what is the apparent magnitude or apparent extenſion of an object but the apparent diſtance of its extremities from one another? and what is the apparent diſtance between two objects in any ſituation or between one object and the ſpectator himſelf, but the apparent extenſion of intermediate objects? And ſince they are ſeldom ſeen alone, excepting through glaſſes, it cannot be doubted, but we eſtimate their diſtances from one another and from our ſelves by our ideas of the magnitudes of thoſe intermediate objects: and every one knows that ſurveyers, gunners, travellers and all ſorts of artificers, who are converſant in meaſuring diſtances, are able to make a truer eſtimate of diſtance by the eye, than others that have not had ſo much experience. Sometimes indeed without attending to thoſe oblique ſurfaces we are ſenſible of the approach of a body by the increaſe of its own apparent magnitude, and on the contrary; and ſometimes we are alſo ſenſible of it when the body is at reſt, provided it be known and familiar to us. For bodies are diſtinguiſhed into ſorts chiefly by their ſhapes and colours, and we reckon them ſmall or great, not in compariſon with bodies of another ſort but with one another; and having found by experience that certain quantities of apparent magnitude of a known body are conſtantly attended by certain quantities of diſtance; the ſenſation of the magnitude of the

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* Anatomy p. 324 3d Edit.

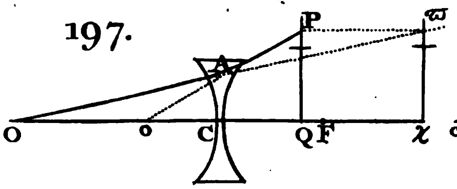
body immediately excites the uſual idea of its diſtance: which is alſo evident in oblique ſurfaces as well as of thoſe that are perpendicular to the eye. For the ideas of variable diſtances muſt either mediately or immediately be excited in the mind by certain variable ſenſations, cauſed by ſome certain variations in the pictures upon the retina. But while the diſtance of the object varies, nothing is varied in its picture, excepting its magnitude; its figure, colour, brightneſs[a] and diſtinctneſs receive no ſenſible variation in moſt caſes: and for one idea to excite another, every one knows it is ſufficient that they have conſtantly been obſerved to go together, as in languages and a thouſand things beſides. Laſtly I have found by abundance of experiments made with glaſſes of all ſorts, that while the apparent magnitude of an object increaſes by moving the glaſs, eye or object, it always appears to approach, and to recede while its apparent magnitude decreaſes, excepting a particular caſe or two to be mentioned hereafter. And theſe experiments ſeem to me to put the queſtion beyond diſpute. For in looking through glaſſes with one eye only, and at a ſingle object, when nothing is perceived in the ſpace interpoſed, how is it poſſible for different apparent magnitudes of the object to ſuggeſt the ideas of different quantities of that inviſible ſpace, according to a certain rule to be mentioned hereafter; if thoſe ideas had not uſually gone together before we looked into the glaſſes? I find alſo that by altering the degrees of apparent brightneſs and diſtinctneſs of an object, either by looking through little holes made with a pin or through lenſes of different figures put cloſe to my eye, or through both at once put cloſe together and to my eye, that neither the apparent magnitude[b], nor apparent diſtance is ſenſibly altered thereby. The reaſon is, we have had no experience in ſuch confuſed viſion with the naked eye, and therefore, though different degrees of confuſion and diſtinctneſs in glaſſes are plainly perceived, yet like the words of an unknown language, their ſignification of diſtance or of any thing elſe is entirely unknown. The ſame may be ſaid of the degrees of brightneſs and obſcurity. By daylight objects appear equally bright at all moderate diſtances from the eye[c], and we retain much the ſame ideas of their diſtances in the night, when we ſee them more obſcurely. The permanent colours and ſhades of bodies ſerve chiefly to diſtinguiſh their apparent ſhapes; and their colours and ſhapes are manifeſt diſtinctions of their various ſorts, but being permanent they are no diſtinctions of their apparent diſtances from the eye. When the eye is fixed and a fixt line is extended from it, the divergency of rays from different points of that line is neither diſtinguiſhed nor ſo much as perceived by ſenſe, by perſons that ſee diſtinctly. It is a rational deduction from ſenſe which informs us that rays diverge from the points of an object; which the majority of mankind are entirely ignorant of: and the ancient philoſophers who thought that ſomething

like

[a] Art. 93.

[b] Art. 109, 117.

[c] Art. 93.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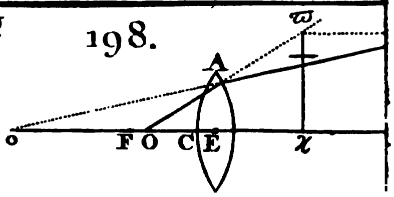
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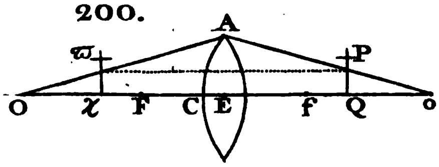
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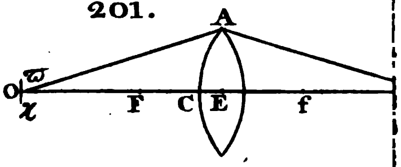
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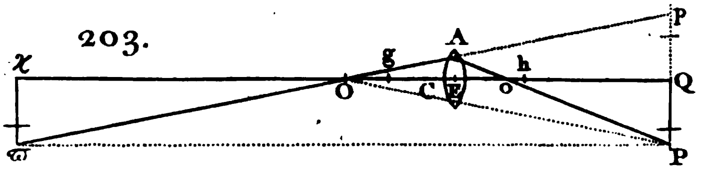
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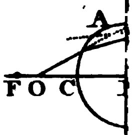

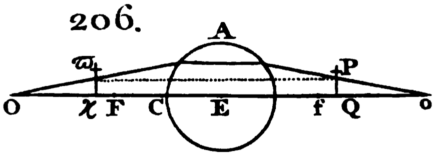
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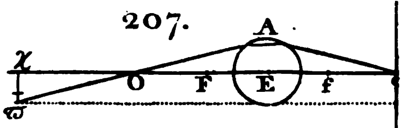
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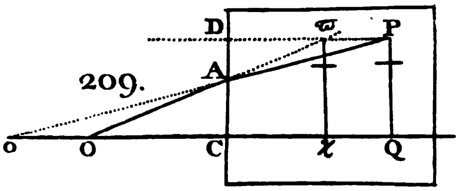
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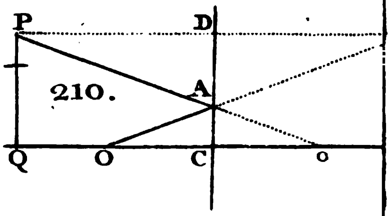
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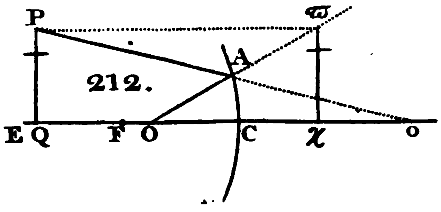
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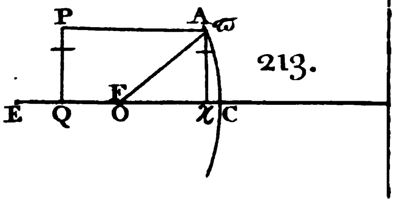
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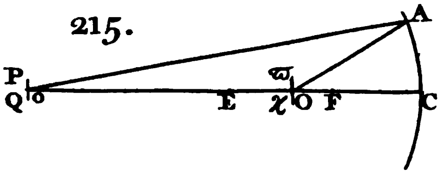
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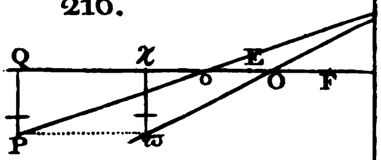
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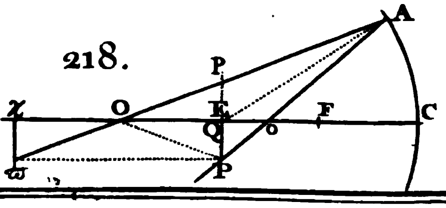
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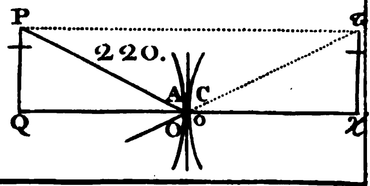
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like rays proceeded from the eye to the object, could diſtinguiſh diſtances as well as we. Therefore the divergency of rays from points at different diſtances is not the medium which introduces the ideas of diſtances into the mind. Sometimes indeed there are degrees of diſtinctneſs and confuſion conſequent upon it, but their relation to diſtance as I ſaid before is not perceived. Beſides this, in viſion with glaſſes, we have ideas of as many different degrees of diſtance conveyed to us, as well when the rays come converging towards points behind the eye as when they diverge from points before it, as will be ſhewn hereafter. The divergency of rays from the place of an object is therefore no cauſe of its appearing in that place. It is alſo matter of fact in painting and perſpective, that our ſenſible ideas of the places of the objects in the picture are quite different from our rational ideas of the places from whence the rays diverge: and the difference in theſe ideas is cauſed by the different apparent magnitudes of the known objects repreſented in the picture. It is alſo evident that our ſenſible ideas of the places of the remoter parts of a long walk or gallery, and of the clouds over head, and of all celeſtial bodies, are quite different from the rational ideas of the places from whence the rays diverge, as will appear more fully hereafter. Neither is diſtance ſuggeſted to the mind by the magnitudes of the angles in a triangle made by the optick axes and the interval between the eyes. For theſe angles are all varied by turning the head ſideways while we look at an object, till at laſt we ſee it at the ſame diſtance with one eye as with both: which ſhews alſo that the faint and confuſed appearance of collateral objects does not alter our ideas of their diſtances. Nor is diſtance ſuggeſted by feeling the turn of the eyes in widening or contracting the interval between pupils, when we direct them to different places. For the place of the object is generally perceived by a ſide-view, before we direct our eyes to view it more diſtinctly. From what has been ſaid it appears to me that the ideas of diſtance are ſuggeſted to the mind by the ideas of the magnitudes of objects.

Apparent diſtance in glaſſes determined. Fig. 197 to 226.

139. Hence it follows that an object ſeen by refraction or reflection, appears at the ſame diſtance from the eye, as it uſually does from the naked eye, when it appears of the ſame magnitude as in the glaſſes. To determine this diſtance in all caſes, I conceive a ray OA to go from the eye at O, and after its laſt reflection or refraction to belong to the focus o, in the common axis OCQ of all the ſurfaces; and to meet an object PQ in P, placed perpendicular to OQ; and that a line $P\varpi$ is drawn parallel to the axis OQ till it meets the ray OA, produced, in ϖ. Then ſuppoſing the object PQ to be removed to the place ϖx, and there to be viewed by the naked eye; ſince it appears under the ſame angle ϖOx or AOC as it appeared under in the glaſſes, when it was at PQ, it will

alſo appear of the ſame magnitude and conſequently at the ſame diſtance from the eye in both caſes[a]. Therefore if when the object is placed at $\varpi\kappa$, its apparent diſtance from the naked eye be repreſented by its real diſtance $O\kappa$, the ſame $O\kappa$ will alſo repreſent its apparent diſtance in the glaſſes when it was at PQ, I ſhall therefore call $O\kappa$ the apparent diſtance of the object PQ, and $\varpi\kappa$ the apparent object.

[a] Art. 138.

When the point P and the ray OA, by which it is ſeen, are on contrary ſides of the axis OQ, the point ϖ and the line $\varpi\kappa$ will be behind the eye, and therefore muſt be viewed by the naked eye inverted and turned about. But if you had rather $\varpi\kappa$ ſhould always be before the eye, in this caſe invert the object PQ, and then ſlide it along the axis; and its extremity P will touch the viſual ray OA, produced, at the ſame diſtance from the eye as before; becauſe the oppoſite angles AOC, $\varpi O\kappa$ are equal.

It varies reciprocally as the apparent magnitude varies.

140. Hence while the eye, object or glaſſes are in motion, the apparent diſtance of the object will increaſe in the ſame proportion as its apparent magnitude decreaſes; and on the contrary. For the ſame apparent diſtance of the ſame object ſeen at $\varpi\kappa$ by the naked eye, varies in that proportion of the ſame apparent magnitude[b].

[b] Art. 99.

Apparent and true diſtance compared in general.

141. Hence alſo the apparent diſtance, $O\kappa$, of an object, PQ, ſeen in glaſſes, is to its apparent diſtance, OQ, ſeen by the naked eye, as its apparent magnitude to the naked eye, to its apparent magnitude in the glaſſes. For conceiving a line OP, which is omitted in the figures for the ſake of ſimplicity; ſince PQ and $\varpi\kappa$ are equal, the former diſtance $O\kappa$, is to the latter OQ, as the latter angle POQ, to the former $\varpi O\kappa$[c].

[c] Art. 60.

Deſign.

The ratio of the apparent and true magnitudes of objects being determined in moſt caſes in the foregoing chapter, their apparent diſtances are alſo determined by this rule. But becauſe this ſubject of apparent diſtance, has hitherto been handled but very imperfectly by all optick writers, it may not be unacceptable to ſome readers to ſee it purſued a little farther. I will therefore deduce all the caſes of apparent diſtance immediately from the definition of it[d], without the help of thoſe former demonſtrations.

[d] Art. 139.

Caſes when equal to each other.

142. The apparent and true diſtances $O\kappa$ and OQ will be equal, firſt when the object touches any thin lens or any ſingle ſurface. For then the points P, A, ϖ will coincide. Secondly, when the eye touches any thin lens or reflecting ſurface. For when the points O, A, C coincide at a lens, the viſual ray will paſs through the middle of it very nearly; and conſequently its incident and emergent parts produced, will be nearly parallel and coincident[e], and ſo the points P, ϖ will nearly coincide: and when the points O, A, C coincide at a reflecting ſurface, the incident and reflected rays produced will make equal angles with the perpendicular $QC\kappa$, and ſo the triangles PCQ, $\varpi C\kappa$ will be equal. Thirdly, when the

Fig. 219, 220.

[e] Art. 42.

Fig. 217.

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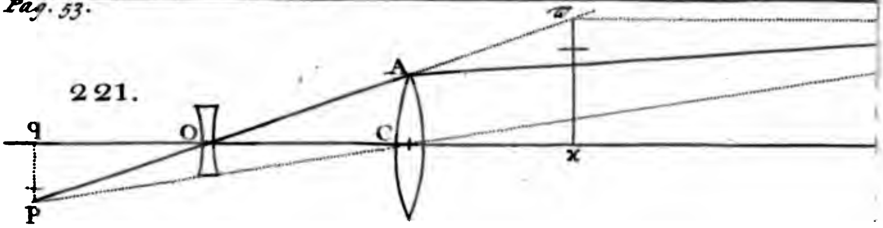

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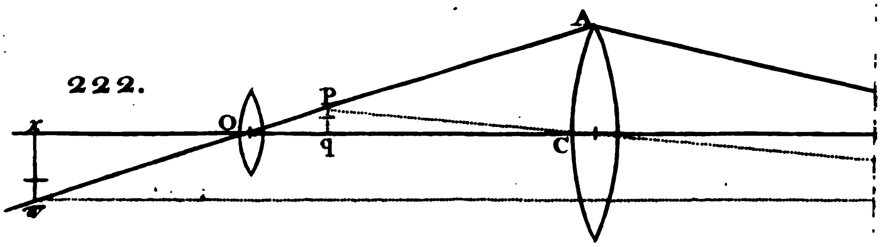

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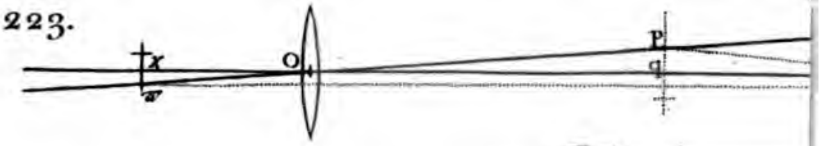

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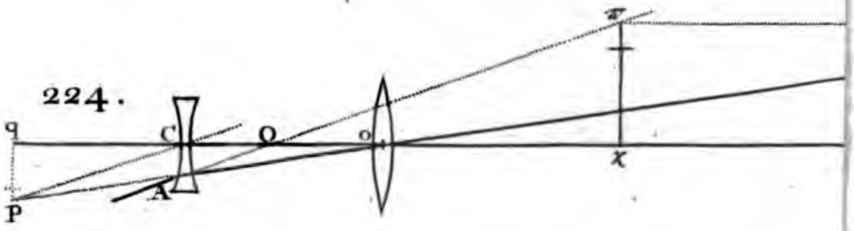

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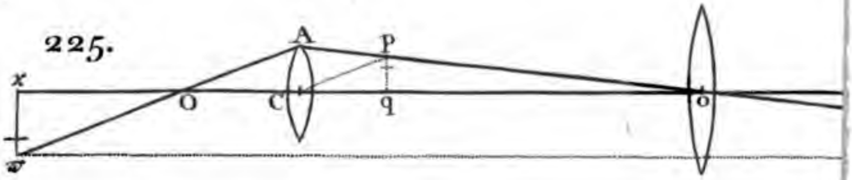

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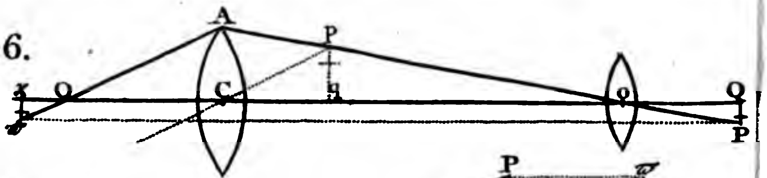

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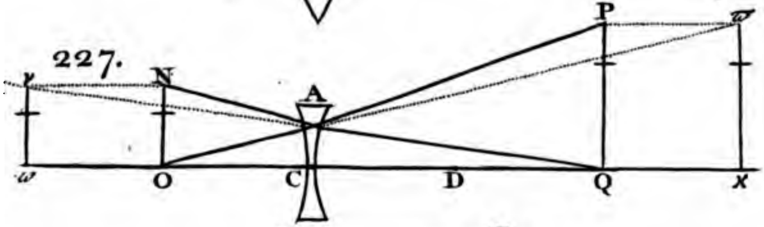

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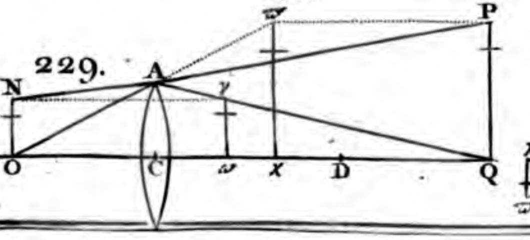

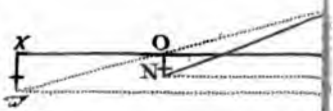

the eye is at the center of a reflecting concave. For then the incident and reflected rays, and conſequently ϖx and PQ, will coincide. Fourthly, when a ray PO coming directly to the eye, makes an angle POQ equal to AOC or ϖOx. For then the triangles POQ, ϖOx are equal. This happens in a reflecting concave when the object is very near its center. For producing the object PQ till it cuts the reflected ray in p, ſince the angles POQ, ϖOx or pOQ are ſuppoſed to be equal, the lines PQ, pQ will alſo be equal, and conſequently a line QA will nearly biſect the angle PAp, when A is very near to C[a]; as a line drawn from the center E will do[b], and ſo the points Q, E are almoſt coincident.

Fig. 203, 207, 218.

a Art. 59.

b Art. 8.

Apparent and true diſtance of the object compared in teleſcopes and microſcopes.

Fig. 221, 222, 223.

143. The apparent diſtance of an object ſeen in a teleſcope or a microſcope, is to its apparent diſtance perceived by the naked eye, as its apparent magnitude to the naked eye, to its apparent magnitude in the teleſcope or microſcope. For conceiving AC to be the object-glaſs, and the eye to be cloſe to the eye-glaſs at O; the viſual ray AO will go in a manner ſtraight through it[c]; and ſo the apparent magnitude and apparent diſtance of the object will continue the ſame as when there was no eye-glaſs: and ſince, when the viſion is diſtinct, the rays in every pencil come parallel through the eye-glaſs, the apparent magnitude and conſequently the apparent diſtance will ſtill continue the ſame as before while the eye is drawn back[d].

c Art. 42.

d Art. 107.

Conſequently the apparent diſtance in a teleſcope is to the apparent diſtance perceived by the naked eye, as the focal diſtance of the eye-glaſs, to the focal diſtance of the object-glaſs; by the 120th and 141ſt articles. Which may thus be demonſtrated independently of the 120th article. Let pq be the image of a remote object terminated by the line PCp; ſo that qC and qO may be the focal diſtances of the object-glaſs and eye-glaſs; then ſuppoſing the object viewed by the naked eye at C, ſince the angles ϖOx, PCQ have equal ſubtenſes ϖx and PQ, the apparent diſtance, Ox, in the teleſcope, is to the apparent diſtance, CQ, to the naked eye at C, as the latter angle PCQ to the former ϖOx[e], or as the oppoſite angle pCq, to the oppoſite angle pOq, or ſince pq ſubtends them both, as the latter focal diſtance qO, to the former qC[e].

e Art. 60.

Fig. 224, 225, 226.

The ſame proportion may be proved when AC repreſents the eye-glaſs of a teleſcope or microſcope, and the object-glaſs is placed at o, the conjugate focus to O. For let pq be the image of the remote object PQ terminated by the line PoA, and when qo and qC are the focal diſtances of the glaſſes at o and C, the ray AO will be parallel to pC[f]. Now the apparent diſtance Ox, is to the apparent diſtance oQ, perceived by the naked eye at o, as the latter angle PoQ to the former ϖOx, or as the oppoſite angle poq to the oppoſite angle AOC or its equal pCq, or as the latter diſtance qC to the former qo.

f Art. 50.

144. An

Apparent diſtances of the object and glaſs compared. Fig. 197 to 226.

144. An object ſeen in any glaſs will appear behind it, or at it, or before it, according as, $\varpi\kappa$ or PQ, the real magnitude of the object is bigger, equal or leſs than AC the part of the glaſs in which it appears. For ſince $\varpi\kappa$ and AC do both ſubtend the ſame or equal angles at the eye, $O\varpi$ will be bigger, equal or leſs than OC, according as $\varpi\kappa$ or PQ is bigger, equal or leſs than AC.

Hence it follows that an object always appears behind any ſurface or glaſs which cannot make rays go parallel that diverged from the eye. For then PQ or $\varpi\kappa$ will always be greater than AC.

The rule is true in a globe or in any number of ſurfaces, taking A for the concourſe of the incident and emergent parts of the viſual ray produced, and a perpendicular AC upon the axis, inſtead of the aperture of a ſingle glaſs.

The diſtances of the real and apparent objects from the glaſs compared. Fig. 209, 210.

145. It appears by the conſtant ſimilitude in the ſhapes of the triangles $A\varpi P$, AOo, that the ratio of $A\varpi$ to AP, that is of the diſtances of the apparent and real objects from the glaſs, is the ſame as the ratio of AO to Ao, that is of the diſtances of the eye and its conjugate focus from the glaſs. Conſequently in refractions at a plane ſurface this ratio is the ſame as of the ſine of incidence to the ſine of refraction [a] of a ray coming from the object to the eye; which in refractions out of water into air is as 3 to 4; and in reflections at a plane it is a ratio of equality [b].

[a] Art. 31.

[b] Art. 23.

Sometimes an object appears in the place of its image.

146. Therefore in theſe two caſes the object appears in the place of its image; not becauſe the rays diverge from that place to the eye, which the ſenſe does not perceive, but becauſe the object is equal to the image; and conſequently its apparent magnitude and diſtance are the ſame as if it was put in the place of the image and viewed there by the naked eye. But if an object be put in the place of an image which is leſs than it ſelf, it will appear bigger [c] and conſequently nearer to the naked eye than it did in the glaſs [d]; that is, the object in the glaſs will appear remoter than the place of its image: and on the contrary. And in general, the apparent diſtance of an object, is to the real diſtance of its laſt image, as the real magnitude of the object, to the real magnitude of that image. Becauſe the apparent object ($\varpi\kappa$) and the laſt image, ſubtend the ſame angle at the eye.

[c] Art. 105.

[d] Art. 138.

How the apparent diſtance varies while the glaſs and eye are fixt and object is moved. Fig. 197 to 220.

147. While the glaſs and eye are fixed, and the object is gradually removed from the glaſs, we may ſuppoſe the lines OA, Ao to be fixt and only the parallel $P\varpi$ to be moveable; and from hence it will be evident, eſpecially from the conſtant ſimilitude in the ſhape of the variable triangle $PA\varpi$, to the fixt triangle oAO, that, in all glaſſes which cannot make the ray AP go parallel to the axis, while AP increaſes, $A\varpi$ and $O\varpi$ will alſo increaſe perpetually, wherever the eye is fixt; and that $O\varpi$ will alſo increaſe perpetually in any other glaſs that can make the ray AP go parallel to the axis, when the eye is fixt between this glaſs

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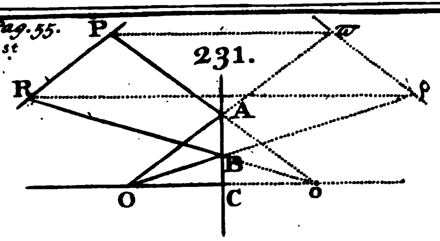

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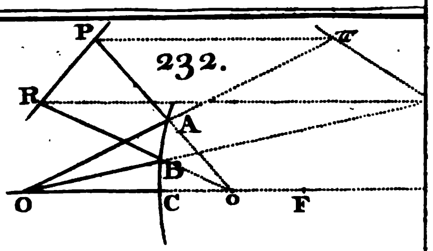

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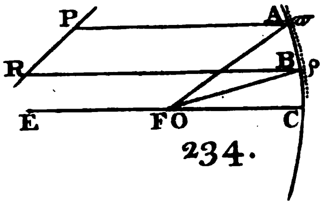

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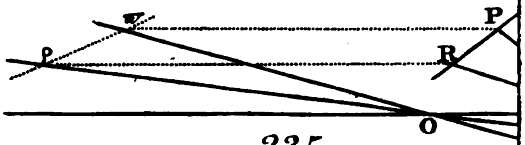

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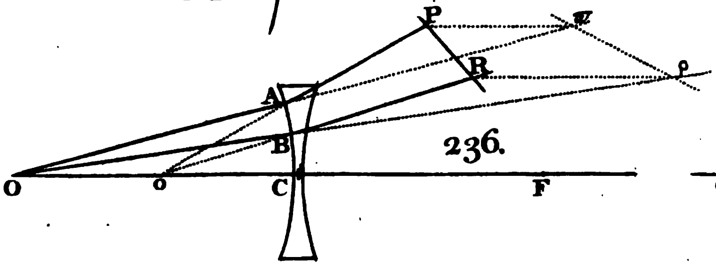

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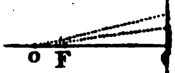

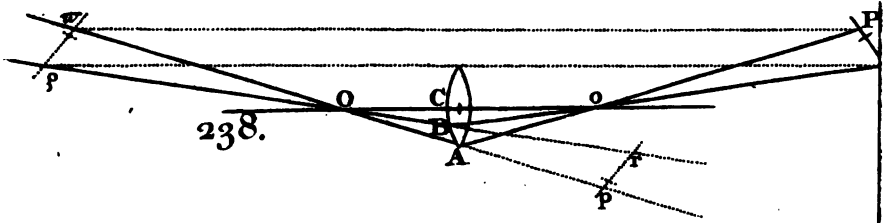

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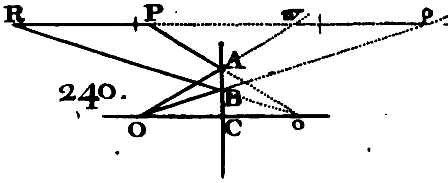

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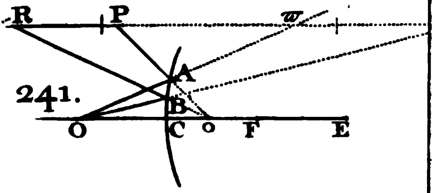

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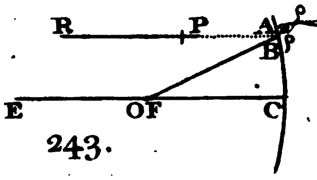

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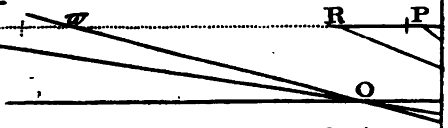

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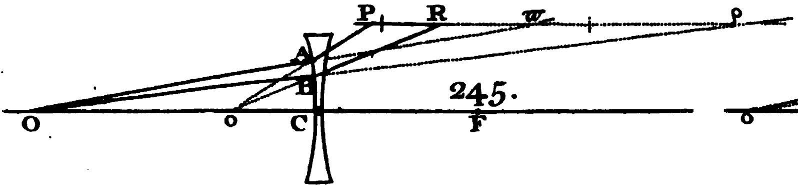

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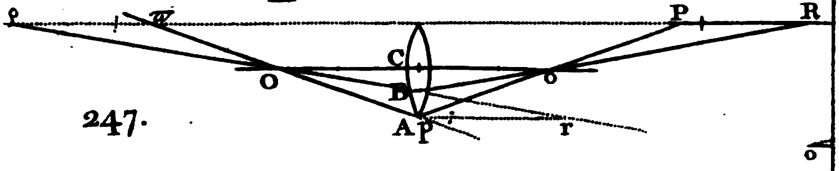

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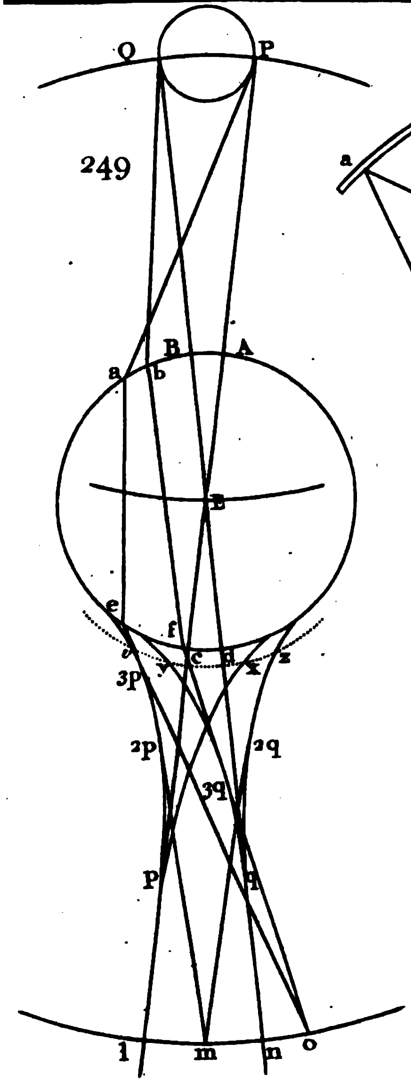
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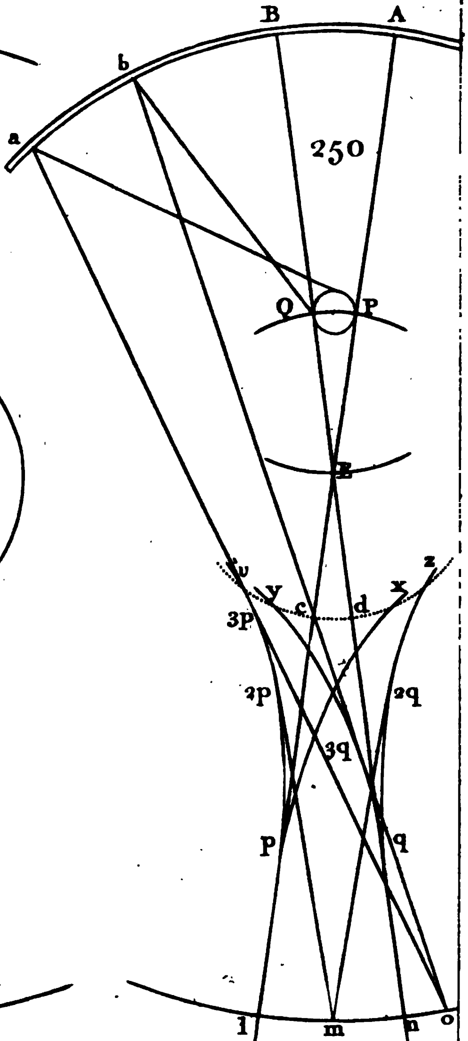
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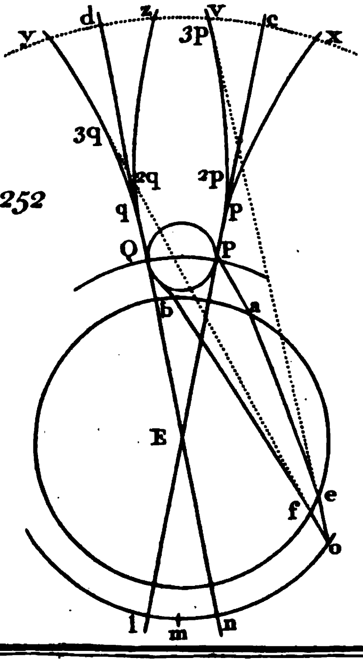
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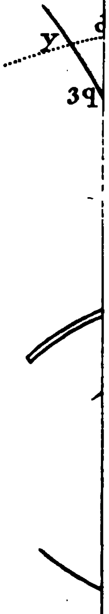

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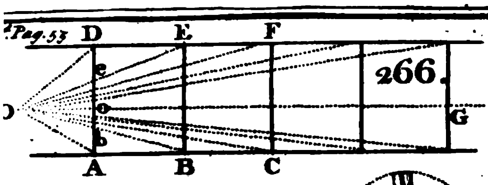
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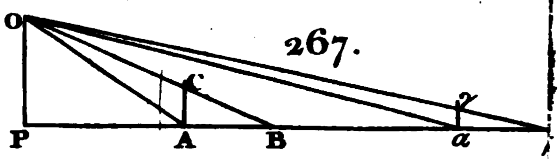
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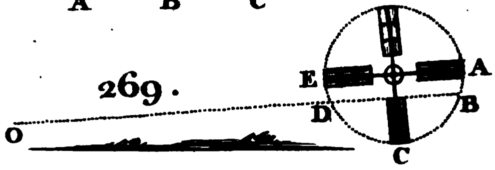
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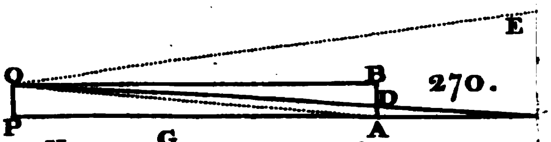
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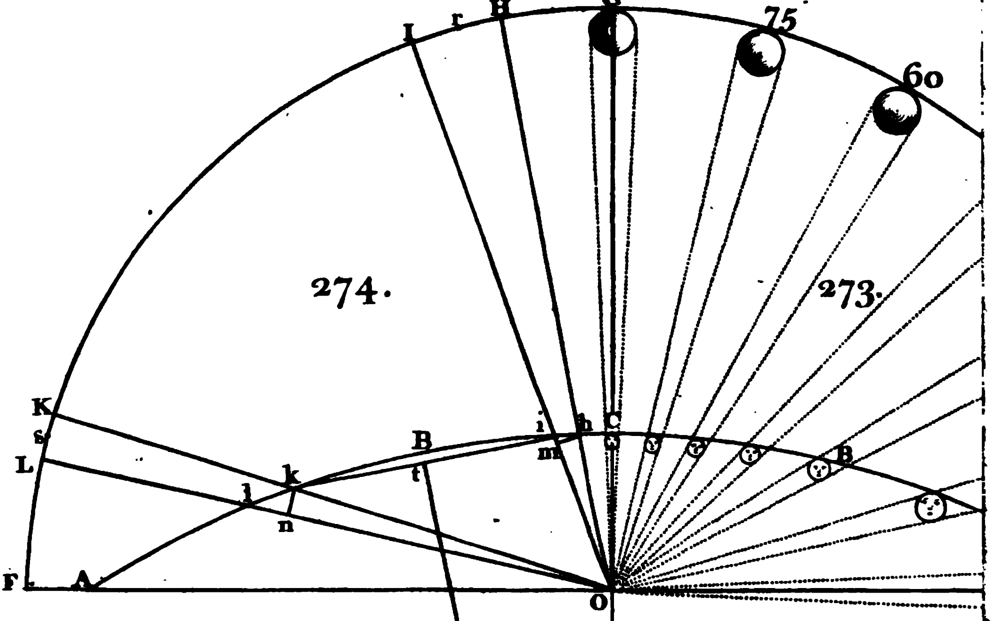
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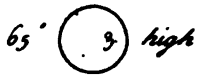

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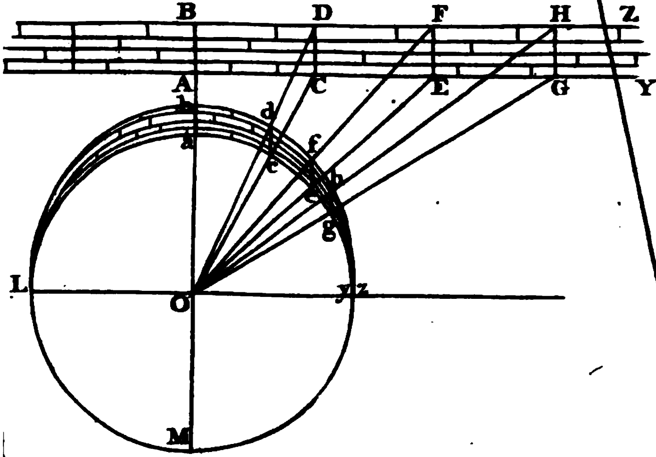

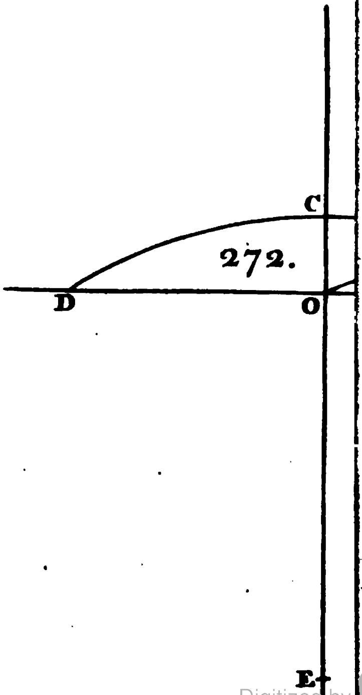
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glaſs and its principal focus. But if the eye be fixt at this focus, $A\varpi$ being nothing, $O\varpi$ will be conſtantly equal to its focal diſtance; and when the eye is fixt on this ſide the focus, $O\varpi$ will decreaſe till P arrives at o, and after it has paſſed over o, $O\varpi$ will increaſe perpetually till it equals OA, when oP equals oA; and alſo till it equals OP, when the angle POQ becomes equal to $\varpi O\kappa$ or AOC; that is when the true and apparent magnitudes of the object become equal.

How it varies when the glaſs and object are fixt and the eye is moved.

148. There will be the like variations of apparent diſtance while the glaſs and object are fixt and the eye is gradually removed from the glaſs; that is in all ſorts of glaſſes and ſurfaces which cannot make diverging rays become parallel, while AO increaſes, $O\varpi$ will alſo increaſe perpetually wherever the object be fixt; and in any other glaſs that can make them parallel, $O\varpi$ will alſo increaſe when the object is fixt between this glaſs and its principal focus. But if the object be fixt at this focus, $O\varpi$ will be conſtantly equal to its focal diſtance; and when the object is fixt on this ſide the focus, $O\varpi$ will decreaſe till o arrives at Q, but after it has paſſed over Q, $O\varpi$ will increaſe perpetually; till it becomes equal to OA and then to OP as in the former article.

See Plate 13.

For the glaſs and object being fixt, the image of the object is alſo fixt in place and magnitude; and being beyond the glaſs in the two firſt caſes, the angle it ſubtends at the eye will continually decreaſe, while the eye recedes from it and from the glaſs; and conſequently the given object $\varpi\kappa$ will ſubtend that decreaſing angle at greater and greater diſtances from the eye: but when the object is at the principal focus, the angle which meaſures its apparent magnitude will be invariable; and conſequently $O\varpi$ will alſo be invariable and equal to the focal diſtance; and when the object is farther from the glaſs than its focal diſtance, its image will be on this ſide of the glaſs, and the eye in receding from the glaſs will firſt come nearer to the fixt image till it paſſes by it, and then recedes from it; ſo that the apparent magnitude will firſt increaſe and then decreaſe, and conſequently the apparent diſtance will firſt decreaſe to nothing, and then increaſe perpetually[a].

Two perſons viewing one another through a lens apppear at the ſame diſtance from each other.

Fig. 227. to 230.

149. Two perſons NO, PQ viewing one another through any given lens AC, appear at equal diſtances from one another. For let two rays PAO, NAQ croſs one another at any point of the lens and let the viſual rays OA, QA produced meet the parallels $P\varpi$, $N\nu$ in ϖ and ν, and the perpendiculars $\varpi\kappa$, $\nu\omega$ will be the apparent objects[b]. Now ſince the bendings of the rays NAQ, PAO are equal[c], the angles NAO, PAQ are alſo equal, and being but ſmall, NO is to PQ (as AO to AQ[d] or) as the angle AQC to the angle AOC[e]: that is the apparent objects $\nu\omega$, $\varpi\kappa$ are proportionable to the angles they ſubtend at the eyes Q, O, and conſequently their diſtances from them are equal, as in all caſes of viſion with the naked eye.

150. Hence

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[a] Art. 140.

[b] Art. 139.

[c] Art. 45.

[d] Art. 57.

[e] Art. 60.

At any two places of the lens equidiſtant from the perſons.

150. Hence when the glaſs is at any two places *C*, *D* equidiſtant from the extremities of the given interval *OQ*, the object will appear at the ſame diſtance to the ſame eye. For *Ox*, the apparent diſtance of the object *PQ*, being equal to, *Qv*, the apparent diſtance of the object *NO* ſeen through the glaſs at the diſtance *QC*, will alſo be equal to the apparent diſtance of the object *PQ*, ſeen through the glaſs removed to the diſtance *OD* equal to *QC*.

How that apparent diſtance varies while the eye and object are fixt and the glaſs is moved.

151. When the interval between the eye and object is fixt and a concave lens is gradually moved from either end of it to the other, the apparent diſtance of the object will firſt increaſe and then decreaſe again; and will be the greateſt of all when the glaſs is exactly in the middle of that interval. But when a convex lens is carried from either end to the other, the apparent diſtance of the object will firſt decreaſe and then increaſe and will be the leaſt of all when the lens is exactly in the middle of the interval, provided it be leſs than 4 times the focal diſtance of the lens; but if it be equal to 4 focal diſtances, the apparent object will ſeem to touch the eye, being infinitely great and infinitely confuſed, when the glaſs is in the middle. And when the interval between the eye and object is bigger than 4 focal diſtances, the object will appear infinitely great and confuſed and conſequently infinitely near to the eye, when the lens is at two places, ſuppoſe at *C* and *D* equidiſtant from the eye and object; ſo that while the lens is carried from either end to the other the apparent diſtance will firſt decreaſe, and then increaſe, till the lens gets to the middle, and then will decreaſe and increaſe again till the lens gets to the end: and when the lens is in the middle the apparent diſtance of the object will be leſs, equal or greater than its true diſtance, according as the whole interval is leſs, equal or greater than 8 focal diſtances of the lens; and conſequently if greater than 8, the apparent diſtance will be equal to the true diſtance, when the lens is at two places between *C* and *D* equidiſtant from them and from the middle: and all this while the apparent magnitude of the object will increaſe when its apparent diſtance decreaſes, and on the contrary[a]; as any one will find that pleaſes to make the experiment. And the reaſon of all theſe appearances is ſhewn by an eaſy rule in the next book.

[a] Art. 140.

Appearances of inclined objects determined. Fig. 231 to 239.

152. When an object *PR* is inclined to the axis of any glaſs, its apparent inclination may be determined, as before[b], by drawing the lines *P*ϖ, *R*ϱ parallel to the axis or unrefracted ray *OC*, till they meet the rays, *OA*, *OB*, by which the points *P* and *R* are ſeen, in ϖ and ϱ; and by drawing the line ϖϱ, which will be the apparent object. Becauſe its extremities ϖ, ϱ, viewed by the naked eye, are the apparent places, in the glaſs, of the extremities of two other objects conceived to touch the extremities of the inclined object *PR*, and to ſtand perpendicular to the axis of the glaſs[c]: obſerving, as before, that when *PR* and *AB* are on con-

[b] Art. 139.

[c] Art 139.

contrary ſides of the axis, the naked eye muſt either be inverted and turned about to view the inclination of $\varpi\varrho$, or elſe two other diſtances *Op* and *Or* muſt be taken equal to $O\varpi$ and $O\varrho$ in oppoſite parts of the ſame rays produced; and then if the glaſs be removed, the line *pr* will appear in the ſame place and poſition as the object *PR* appeared in, in the glaſs.

And of objects parallel to the axis.
Fig. 240 to 248.

153. Hence if an object *PR* be parallel to the axis of the glaſs, produce it till it cuts the viſual rays *OA*, *OB* in ϖ and ϱ; and the line $\varpi\varrho$ will appear to the naked eye in the ſame place and poſition as *PR* appeared in, in the glaſs. It is to be obſerved that though the real places of the lines *PR*, $\varpi\varrho$ are parallel to the axis, yet they do not appear ſo to the naked eye; but ſeem to converge towards the remoter parts of the axis, for a reaſon to be mentioned in Art. 156.

Definition.
Fig. 131. &c.

By the deſcription of cauſticks in Art. 69 &c. and by the figures there explained, it is eaſy to underſtand, that the edge of a thin plate may be formed into a curve of ſuch a ſhape and degree of convexity, that when it is applyed within the concavity of a leg of a given cauſtick, it ſhall touch every ray in a different point of its convexity *. This convex edge it ſelf may be called the leg of the cauſtick; and is repreſented in the following figures by the curve $p\,3pv$ or $q\,3qy$, &c.

* See Art. 445, 446.

Viſual rays are tangents to the leg of a cauſtick.
Fig. 249 to 253.

154. Hence when the eye is fixt at any point *o*, placed any where but in the very curve of a given cauſtick, formed by refractions or reflections of all the rays that flowed from *P*; the viſual ray * by which the point *P* is then ſeen, may be found by drawing a line from the eye at *o*, that ſhall touch a leg of the cauſtick in a ſingle point $3p$, without cutting the ſame leg. And if you conceive a fine line or ſtring to be fixt at the fartheſt end of this leg from the eye, and to be lapped upon a part of its convexity, and thence to be extended from it in a ſtraight line $3po$, the point *P* will always appear in the ſucceſſive directions of this ſtring, while the eye, or the cauſtick it ſelf, is moved ſideways.

* Art. 90.

How the apparent magnitude of an object varies by moving the eye, object or glaſs ſideways.

155. If a ſmall round object be ſeen inverted through a ſphere, or in ſo large a portion of a reflecting concave ſurface as to form a cauſtick, it will appear biggeſt and neareſt to the eye placed in a line drawn through the object and the center of the ſphere or ſpherical ſurface: and the object will appear gradually ſmaller and remoter while the eye, or the ſphere, or the object it ſelf, is moved ſideways: and the contrary appearances may happen in a leſſer degree when the object is ſeen upright, in the caſes ſpecified by the figures, which among other ways may be thus demonſtrated.

Fig. 249 to 253.

Through the center *E* of the ſphere or ſpeculum, draw two lines *EP*, *EQ* touching the oppoſite ſides of the ſmall round object *PQ*; and of all the rays that flow from *P*, let the neareſt to the line *PEp*, belong to the focus *p* after refractions or reflections; and the reſt of them will be formed into a cauſtick, whoſe legs *pv*, *px* are always convex towards *PEp*, the axis of the pencil *. Alſo let *qy* and *qz* be the legs of another cauſtick formed by the

* Art. 69 &c.

pencil flowing from Q, whoſe axis is QEq. Then with the center E, and any ſemidiameter El, draw an arch $lmno$, cutting the (produced) axes Pp, Qq in l and n; and from the eye firſt placed at m within the angle lEn under the axes, draw the lines $m2p$, $m2q$ ſo as to touch a leg of each cauſtick; and the object PQ will appear under the viſual angle $2pm2q$*; Again from the eye removed to any point o placed out of the angle lEn under the axes, draw two other lines $o3p$, $o3q$ touching a leg of each cauſtick at $3p$ and $3q$; and the object PQ will now appear under the viſual angle $3po3q$*,

* Art. 154.

* Art. 154.

Now while the eye is moving ſideways in the arch mno, one of the points of contact $2p$ will move continually in the ſame leg from $2p$ to $3p$; but the other point $2q$ will move firſt from $2q$ to q in the ſame leg, and then will return along the other leg of the ſame cauſtick from its cuſp q to the point $3q$. And ſo the viſual rays $3po$, $3qo$ will now come to the eye at o, from thoſe legs of the two cauſticks that lye both on the ſame ſides of their axes Ep, Eq.

Let any circle deſcribed upon the center E, cut the two laſt mentioned legs $3pp$, $3qq$, in the points v and y, and their reſpective axes Ep, Eq in c and d; and ſince the two cauſticks, upon account of the equal diſtances EP, EQ, reſulting from the roundneſs of the object PQ, are equal to each other, it is eaſy to underſtand that the arch cv is equal to dy, and conſequently that the arch vy between theſe two legs, will be equal to the correſponding arch cd between their axes. And the like property being true in every circle deſcribed about the ſame center E, it appears that theſe legs approach towards each other in approaching towards the center E.

Fig. 249, 250.

Therefore when the object appears inverted to the eye any where in the arch mno*, it will appear biggeſt when the eye is between the axes Ep, Eq, and gradually ſmaller, and conſequently remoter*, while the eye is moving ſideways, becauſe the viſual angle decreaſes. And it is not difficult to underſtand by conſidering the 251ſt, 252d, and 253d figures, that the contrary may happen when the object appears upright.

* Art. 103.

* Art. 138.

It is eaſy to apprehend that there will be a like variety of appearances, when the eye ſtands ſtill and the object is moved ſideways in a circle PQ whoſe center is E; and alſo when the center E of the ſphere or ſpeculum, is moved ſideways in a circle whoſe center is at the object,

Parallel lines ſeen obliquely appear to converge.

Fig. 266.

156. Parallel lines ſeen obliquely, as ABC, DEF, appear to converge more and more as they are farther extended from the eye. Becauſe the apparent magnitudes of their perpendicular intervals AD, BE, CF, &c. are perpetually diminiſhed. And for the ſame reaſon they appear to converge towards an imaginary line OG drawn from the eye parallel to them.

This is the reaſon that the remoter parts of a walk or a floor appear to aſcend gradually, and the cieling to deſcend towards the horizontal line OG: and that the ſurface of the ſea, ſeen from an eminence, appears to aſcend gradually in going from the ſhore; and that the upper parts of very high buildings ſeem to lean forward over the eye below; becauſe they ſeem to approach towards a vertical line OG.

How the apparent magnitude of an oblique object varies. Fig. 267.

157. The apparent magnitude of a given line, AB, ſeen very obliquely at a given diſtance, OA, increaſes and decreaſes in proportion to the increaſe and decreaſe of, OP, the perpendicular diſtance of the eye from the line AB produced; provided the diſtance AO be very large in compariſon to AB. For let the ray BO cut a line AC perpendicular to AB in C; and while the eye is raiſed or depreſſed in the perpendicular OP, the line AC will increaſe and decreaſe as OP does, and ſo will the angle AOC ſubtended by AC[a], and this angle meaſures the apparent magnitude of AB[b].

Hence the apparent magnitudes of equal parts AB, $\alpha\beta$ of a line $PA\alpha$, ſeen very obliquely at great diſtances from the eye, are reciprocally in a duplicate proportion of thoſe diſtances. For example let $O\beta$ be double of OB, and the angle OBP will be double of $O\beta P$[c], and accordingly ſince AB, $\alpha\beta$ are equal, the perpendicular AC will be double of $\alpha\gamma$ and being ſeen twice as near as $\alpha\gamma$, will appear four times bigger than $\alpha\gamma$. Again if $O\beta$ be treble of OB, the line AC will be treble of $\alpha\gamma$, and being ſeen three times nearer than $\alpha\gamma$, will appear nine times bigger than $\alpha\gamma$; and ſo on.

Hence the apparent intervals between a row of columns are diminiſhed in a greater proportion than their apparent heights.

Why unequal diſtances appear equal.

158. This quick diminution of the apparent magnitudes of the remoter parts of long lines or diſtances, is the cauſe of great difficulty and uncertainty in our eſtimate of their quantities. For be the differences of ſeveral diſtances or heights never ſo great in themſelves, they will become inviſible at laſt by reaſon of the ſmalneſs of the angles they ſubtend at the eye, occaſioned by their obliquity: and then thoſe unequal heights and diſtances will appear equal.

And ſometimes ſhorter ſometimes longer.

159. Diſtances from the eye ſeen upon a rough, uneven ſurface appear ſhorter, than if it was perfectly plane. For the inequalities of the ſurface, ſuch as hills and holes and rivers that lye low and out of ſight, either do not appear or hinder the parts from appearing that lye behind them; and ſo the whole apparent diſtance is diminiſhed by the parts that do not appear in it. It is a common obſervation that the banks of a river appear contiguous to a diſtant eye when the river is low and is not ſeen: inſomuch that travellers in a ſtrange place are frequently uncertain where the river runs, and whether the objects they ſee before them are on this ſide or on that ſide of it. And when a flag or weather-cock appears above any high building, the ſight alone at a moderate diſtance cannot diſtinguiſh whether it belongs to that building or to ſome other

[a] Art. 59.

[b] Art. 98.

[c] Art. 60.

behind it. In like manner the ſun, moon, and clouds, and the tops of mountains, and all objects in the horizon, when ſeen in the ſame direction appear all at the ſame diſtance.

Fallacies in viſion.

160. The four laſt articles afford a ſolution of ſeveral fallacies in viſion, ſome of which I have here collected. Since oblique diſtances appear longer in proportion as the eye is raiſed higher, to view them more fully; it follows that being placed at a diſtance from a gentle aſcent, like the ſtage at a play-houſe, or a riſing mount at the end of a walk, we ſhall judge thoſe aſcents much longer than if they were level; eſpecially if they be artfully contracted in the remoter parts. For by not obſerving, or attending to, the reality of theſe aſcents, we form the ſame idea of them, as is uſually ſuggeſted to the mind by a longer level walk with parallel ſides. Now ſince the riſing of the ground together with a gradual diminution of its breadth, when not obſerved, do make it ſeem longer and conſequently leſs diminiſhed in breadth than if the ſame extent was level, with parallel ſides; it ſhould follow that a gentle aſcent alone, whoſe ſides are parallel, ſhould ſtill widen the appearance of their remoter parts, ſo as to make the parallel ſides appear parallel or even diverging; which is contrary to the common appearance of parallel ſides. A deception of this kind may be ſeen in a Viſta of parallel rows of trees, when viewed at the front of the Honourable Mr. *North*'s houſe at *Rougham* in *Norfolk*; as I am informed by a neighbour of his, my worthy friend *Martin Folkes* Eſq; whoſe great knowledge and curioſity lets nothing eſcape him. Being aſſured by Mr. *North* that the trees were parallel, which ſeemed to diverge, he was much ſurprized at this uncommon appearance; till after a little conſideration, he perceived that the cauſe of the deception was a gentle riſing of the ground where the trees were planted, and a gentle deſcent for half a mile from the houſe to the beginning of the plantation.

He has alſo told me, that upon coming into a ſtreet, in a dark night, where there was but one row of lamps, he has often miſtaken the ſide of the ſtreet they were in; which he accounts for in this manner. Let *O* be the ſpectator, *A*, *B*, *C*, *D*, the lamps on his right hand; *AaO*, *BbO*, *CcO*, *DdO* the rays that come to his eye. Now if he happens to imagine the neareſt lamp *A* to be the remoteſt, ſuppoſe at *a*; he will conſequently imagine all the reſt to be at *a*, *b*, *c*, *d* in a contrary ſituation of a line extended on his left hand. Fig. 268.

The oblique ſituation of an object ſeen alone is ſuggeſted to the mind by a greater apparent magnitude or a diſtincter perception of the nearer than of the remoter parts. And conſequently if the object be ſo remote or ſo uniform that we are not affected with a ſenſible difference in thoſe perceptions, we are ſubject to miſtake its poſition. For an object may appear under the ſame angle *AOD* in two oblique poſitions *AD* and *ad*. Fig. 268.

Hence we ſometimes miſtake the poſition of a weather-cock or a flag; and by taking the neareſt end of the ſail of a wind-mill for the remoteſt, we ſometimes miſtake the courſe of its circular motion. For if a ſpectator at O, ſituated nearly in the plane of the ſails produced, imagines the fartheſt end A of a ſail AE to be the neareſt, and the real motion of the ſails be in the order of the letters $ABCDE$; when A is moved to B and the line BO is drawn, cutting the circle $ABCDE$ in D; ſince he firſt imagined the end A to be at E, he will not now conceive it at B but at D; and ſo will imagine the courſe of the motion to be from E to D; which is contrary to the real motion from A to B. The uncertainty we ſometimes find in the courſe of the motion of a branch or hoop of lighted candles, turned round at a diſtance, is owing to the ſame cauſe: and alſo that we miſtake a convex for a concave ſurface ſometimes with the naked eye, but more frequently in viewing ſeals and impreſſions with a convex glaſs or a double microſcope; and hills and valleys in the moon with teleſcopes, eſpecially if they invert the object: being led into the miſtake by an imperfect judgment of the diſtances of the parts of the object, and confirmed in it by a contrary poſition of the ſhadows caſt by a ſide light.

Fig. 269.

We are frequently deceived in our eſtimates of diſtance by any extraordinary magnitudes of objects ſeen at the end of it: as in travelling towards a large city or a caſtle or a cathedral church or a mountain larger than ordinary, we think they are much nearer than we find them to be upon trial. For ſince by experience the ideas of certain quantities of known diſtances are uſually annexed to the apparent magnitudes of known objects of a common ſize; and ſince the apparent magnitudes of thoſe larger objects at a greater diſtance are the ſame as of the ſmaller at a ſmaller diſtance, it is no wonder they ſuggeſt the uſual idea of ſmaller diſtance annext to more common objects. This is farther evident, becauſe we are ignorant of the nature of the country; and of the inequalities in the ground interpoſed.

Animals and all ſmall objects ſeen in valleys, contiguous to large mountains, appear extraordinary ſmall; becauſe we think the mountain is nearer to us than if it was ſmaller; and we ſhould not be ſurprized at the ſmalneſs of the neighbouring animals if we thought them farther off. In like manner when they are placed upon the top of the mountain or upon a large building and are viewed from below, we think they are extraordinary ſmall for the ſame reaſon, and alſo becauſe we judge the mountain or the building to be lower in proportion than if it was ſmaller; both becauſe of its extraordinary magnitude and greater obliquity of its higher parts to the viſual rays. *Dechales* tells us that while he ſtood at the bottom of a mountain, he once obſerved a parcel of crows going to fly over it, which at firſt he thought were higher than the mountain; becauſe, I ſuppoſe, they appeared ſo very ſmall in compariſon to it; but he found they ſpent half an hour in aſcending before they got to the top of it [1]. The part of the *Monument* extant above the tops of the adjoining houſes, I am told, is 5 times longer than the height of the houſes, and yet from below that part appears but two or three times longer at moſt; becauſe of its unuſual magnitude and obliquity to the ſight.

Aguilonius mentions a fallacy in diſtance which he had frequently obſerved and admired. In a warm ſummers morning when fogs are exhaled from moiſt ground, we frequently ſee them very near us in ſome known place; but ſo ſoon as they are ſeparated from the ground and are going to aſcend they appear ſo remote, that, ſays he, I could never have believed they hung over that place, had I not ſeen them there but the moment before [2]. The reaſon is they then appear in the manner and direction of other remote clouds in the horizon; whoſe difference in diſtance cannot be diſcerned, for want of ſome viſible ſurface extended between them; like the ſurface of the ground when the riſing cloud lay upon it.

It is ſaid to be a common obſervation made by travellers in the night or the dusk of the evening, that near objects, as trees and houſes are often taken to be very large and remote. The reaſon may be, that being unable to diſcern the quantity of ground interpoſed, they refer them to the brighter ſky in the horizon, and ſo think they are remoter and conſequently larger: as I remember a red coat of arms, upon the top of an iron gate at the end of a walk, was taken for a brick houſe in the fields beyond it.

The greateſt quantity of apparent diſtance determined. Fig. 270.

161. If the ſurface of the earth was perfectly plane, the diſtance of the viſible horizon from the eye would ſcarce exceed 5000 times the height of the eye above the ground, or the diſtance of 5 miles ſuppoſing the height of the eye between 5 and ſix foot: and all objects placed beyond this diſtance would appear in the viſible horizon. For let *OP* be the height of the eye above the line *PA* drawn upon the ground; and if an object *AB*, equal in height to *PO*, be removed to a diſtance *PA* equal to 5000 times that height, it will hardly be viſible by reaſon of the ſmalneſs of the angle *AOB* [a]. Conſequently any diſtance *AC*, how great ſoever, beyond *A*, will be inviſible. For ſince *AC* and *BO* are parallel, the ray *CO* will always cut *AB* in ſome point *D* between *A* and *B*; and therefore the angle *AOC* or *AOD* will always be leſs than *AOB*, and therefore *AD* or *AC* will be inviſible. Conſequently all objects and clouds, as *CE* and *FG*, placed at all diſtances beyond *A*, if

[a] Art. 97.

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[1] *Dechales* Curſus Math. Tom. 3. p. 435.

[2] *Aguilonii* Optica p. 223.

they be high enough to be viſible or to ſubtend a bigger angle at the eye than *AOB*, will appear at the horizon *AB*; becauſe the diſtance *AC* is inviſible.

Apparent concavity of the ſky explained. Fig. 271.

162. Hence if we ſuppoſe a vaſt long row of objects, or a vaſt, long wall, *ABZY*, built upon this vaſt plane, and its perpendicular diſtance, *OA* from the eye at *O*, to be equal to, or greater than, the diſtance *Oa* of the viſible horizon, it will not appear ſtraight but circular, as if it was built upon the circumference of the horizon *acegy*: and if the wall be continued to an immenſe diſtance, its extream parts *YZ* will appear in the horizon at *yz* where it is cut by a line *Oy* parallel to the wall. For ſuppoſing a ray *YO*, the angle *YOy* will become inſenſibly ſmall. Imagine this infinite plane *OAYy*, with the wall upon it, to be turned about the horizontal line *Oy*, like the lid of a box, till it becomes perpendicular to the other half of the horizontal plane *LMy*, and the wall parallel to it, like a vaſt ceiling over head; and then the wall will appear like the concave figure of the clouds over head. But though the wall in the horizon appeared in the ſhape of a ſemicircle, yet the ceiling will not, but much flatter. Becauſe the horizontal plane was a viſible ſurface, which ſuggeſted the idea of the ſame diſtances quite round the eye, but in the vertical plane extended between the eye and the ceiling, there is nothing that affects the ſenſe with an idea of its parts but the common line *Oy*, conſequently the apparent diſtances of the higher parts of the ceiling will be gradually diminiſhed in aſcending from that line. Now when the ſky is quite overcaſt with clouds of equal gravities, they will all float in the air at equal heights above the earth, and conſequently will compoſe a ſurface, reſembling a large ceiling, as flat as the viſible ſurface of the earth. Its concavity therefore is not real but apparent: and when the heights of the clouds are unequal, ſince their real ſhapes and magnitudes are all unknown, the eye can ſeldom diſtinguiſh the unequal diſtances of thoſe clouds that appear in the ſame directions, unleſs when they are very near us, or are driven by contrary currents of the air. So that the viſible ſhape of the whole ſurface remains alike in both caſes. And when the ſky is either partly overcaſt or perfectly free from clouds, it is matter of fact we retain much the ſame idea of its concavity as when it was quite overcaſt. But if any one thinks that the reflection of light from the pure air, is alone ſufficient to ſuggeſt that idea, I will not diſpute it.

Apparent concavity of the ſky determined.

163. The concavity of the heavens appears to the eye, which is the only judge of an apparent figure, to be a leſs portion of a ſpherical ſurface than a hemiſphere: I mean that the center of the concavity is much below the eye, and by taking a medium among ſeveral obſervations, I find the apparent diſtance of its parts at the horizon is generally between three and four times greater than the apparent diſtance of its

parts

Fig. 272.

parts overhead. For let the arch *ABCD* repreſent the apparent concavity of the ſky, *O* the place of the eye, *OA* and *OC* the horizontal and vertical apparent diſtances, whoſe proportion is required. Firſt obſerve when the ſun or the moon or any cloud or ſtar is in ſuch a poſition at *B*, that the apparent arches *BA*, *BC*, extended on each ſide of this object towards the horizon and zenith, ſeem equal to the eye; then taking the altitude of the object *B* with a quadrant, or a croſs-ſtaff, or finding it by aſtronomy from the given time of obſervation, the angle *AOB* is known. Drawing therefore the line *OB* in the poſition thus determined, and taking in it any point *B* at pleaſure, in the vertical line *CO* produced downwards ſeek the center *E* of a circle *ABC*, whoſe arches *BA*, *BC*, intercepted between *B* and the legs of the right angle *AOC*, ſhall be equal to each other; then will this arch *ABCD* repreſent the apparent figure of the ſky. For by the eye we eſtimate the diſtance between any two objects in the heavens by the quantity of ſky that appears to lye between them; as upon earth we eſtimate it by the quantity of ground that lyes between them. The center *E* may be found geometrically by conſtructing a cubick equation, or as quick and ſufficiently exact by trying whether the chords *BA*, *BC*, of the arch *ABC* drawn by conjecture, are equal; and by altering its radius *BE* till they are ſo. Now in making ſeveral obſervations upon the ſun and ſome others upon the moon and ſtars, they ſeemed to me to biſect the vertical arch *ABC* at *B*, when their apparent altitudes or the angle *AOB* was about 23 degrees; which gives the proportion of *OC* to *OA* as 3 to 10 or as 1 to $3\frac{1}{3}$ nearly. When the ſun was but 30 degrees high, the upper arch ſeemed always leſs than the under, and I think always greater when the ſun was about 18 or 20 degrees high.

Why the ſun and moon appear bigger near the horizon than higher up.
Fig. 273.

164. I have been the more particular in conſidering the apparent figure of the ſky, becauſe I do not find it has ever yet been determined; although it be abſolutely neceſſary to a ſatisfactory ſolution of ſeveral noted appearances in the heavens. For inſtance, ſuppoſing the arch *ABC* to repreſent that apparent concavity, I find the diameter of the ſun or moon will ſeem to be greater in the horizon than at any propoſed altitude, meaſured by the angle *AOB*, in the proportion of its apparent diſtances *OA*, *OB*. The numbers that expreſs theſe proportions are ſet down in this table, over againſt the correſponding altitudes of the ſun or moon, and are alſo exactly repreſented to the eye in the 273d figure, in which the moons placed in the quadrantal arch *FG*, deſcribed about the center *O*, are all equal to each other, and repreſent the body of the moon at the heights there noted;

The ſun or moon's altitudes in degrees.	Apparent diameters or diſtances.
00	100
15	68
30	50
45	40
60	34
75	31
90	30

and the unequal moons in the concavity ABC are terminated by the viſual rays that come from the circumference of the real moon, at thoſe heights, to the eye at O. The diameters of theſe unequal moons at A and B do therefore bear the ſame proportion to each other as their apparent diſtances OA, OB[a]; and they muſt appear in the very ſame proportion that they really have in this concave, becauſe we judge all objects in the heavens to be in this very ſurface[b]: and ſo the appearance to the eye is exactly the ſame as if ſeveral moons were painted upon a real ſurface ABC in the proportions here aſſigned; in which caſe we ſhould certainly judge the real magnitudes of the larger paintings of the lower moons to be really larger; though the viſible magnitudes of them all, anſwering to their equal images upon the retina, were exactly equal. And let any one conſider when he looks at the moon, whether it does not ſeem to be a real, tangible ſubſtance, as well as all other objects we look at.

[a] Art. 57.

[b] Art. 161, 162.

This ſolution confirmed by obſervations of ſtars.

165. For the ſame reaſon all other objects, and diſtances of ſtars in the heavens, as well as the ſun and moon, muſt ſeem to be greater in the horizon than in higher ſituations; and it is well known they do ſo. Hence I deduce another experimental proof, that theſe proportions of the moon's apparent magnitudes are exactly aſſigned. In a clear ſtar-light night take notice of the diſtance of any two ſtars that lye very near each other, and as high as poſſible; and at the ſame time pick out two other ſtars ſituated as low as poſſible, whoſe diſtance from each other ſeems equal to that of the two higher ſtars. Then by a globe, map, or by calculation, find the real diſtances of each pair of ſtars in degrees and minutes and alſo the altitudes of the middle points of thoſe diſtances above the horizon; and take the arches Fr, Fs equal to them; and ſet off the arches rH and rI each equal to half the diſtance of the higher ſtars, and likewiſe sK and sL each equal to half the diſtance of the lower. Then from O to the points H, I, K, L draw lines cutting the concave ABC, already determined by the method of biſection[c], in the points b, i, k, l. Theſe points, if each couple of ſtars were in a vertical circle, would be their apparent places; and if their ſituation be not vertical, yet the perpendicular ſubtenſes bm, kn (of the angles bOi, kOl) which in ſtars very near together are the meaſures of their apparent diſtances, are not altered by their oblique ſituation. Now by ſeveral obſervations and conſtructions I have found theſe ſubtenſes bm, kn to be nearly equal to one another; and ſince they appeared ſo in the heavens, it ſhews that the concave ABC was determined right. Therefore if HI and KL were the real diameters of two unequal moons, they would appear equal at b and k, and conſequently if the lower moon at KL be increaſed till it equals the higher at HI, then the angles kOn, bOm being now equal, the

Fig. 274.

[c] Art. 163.

ſubtenſe kn will be greater than hm in the ratio of their apparent diſtances Ok to Oh[a], which is the thing we aſſerted of the real moon.

a Art. 57.

Figure of the ſky determined another way.

166. Hence we have another method for finding the apparent figure of the heavens by the foregoing obſervations of the ſtars. Aſſume one of the diſtances from the eye, as Ok, of any length you pleaſe, and take the other Oh in proportion to Ok as KL to HI, and joining hk biſect it in t, and draw a line tE perpendicular to it till it meets CO, produced, in E; this will be the center of the apparent concave; which I need not ſtay to demonſtrate. In the diſtances HI, KL, eſpecially in the latter, if the two ſtars be in a vertical circle or near it, allowance ſhould be made for the refraction of the air.

The like appearances ſolved in the rain-bow and halos.

167. This apparent concave, being leſs than a hemiſphere, is alſo the cauſe that the breadths of the colours in the inward and outward rainbows, and the interval between the bows, appear leaſt at the top and greateſt at the bottom, and in deſcending from top to bottom are gradually increaſed, though the angles ſubtended at the eye by all thoſe breadths are the ſame in every part of the bows: and by an eſtimate of the apparent breadths of the inward rain-bow at two different heights, made by a friend, I determined the apparent concavity of the ſky to be much the ſame as by the former methods. And I take it to be owing to the ſame cauſe that a halo about the ſun or moon does not appear circular and concentrick to the ſun or moon but oval and excentrick, with its longeſt diameter perpendicular to the horizon, and extended from the moon further downwards than upwards, as Sir *Iſaac Newton* has deſcribed one in his opticks, page 290. For it appears by *Hugen*'s theory of halos, explained hereafter, that the rays which cauſe their viſible appearance compoſe the ſurface of a cone, whoſe ſection made by a plane perpendicular to the ray that comes to the eye from the ſun or moon, is circular and concentrick to the ſun or moon; and therefore an oblique ſection of it made, as it were, by the apparent concavity of the ſky, which is the ſame as the perſpective projection of it upon that concave, muſt be ſuch an oval figure as Sir *Iſaac Newton* has deſcribed. This oval figure is alſo taken notice of in a halo obſerved by Mr. *Whiſton*[1], and its excentricity has been obſerved by Dr. *Halley*[2]: and I have lately obſerved the ſame things my ſelf even when the moon was very high[3]. Now ſince the angle which the diameter of a halo ſubtends at the eye is always obſerved to be 45 or 46 degrees, I reckon that when the bottom of the halo is near the horizon, and conſequently its apparent figure is moſt oval, the apparent vertical diameter ab is divided by the moon in the proportion of about 2 to 3 or 4, and is to the horizon-

Fig. 275.

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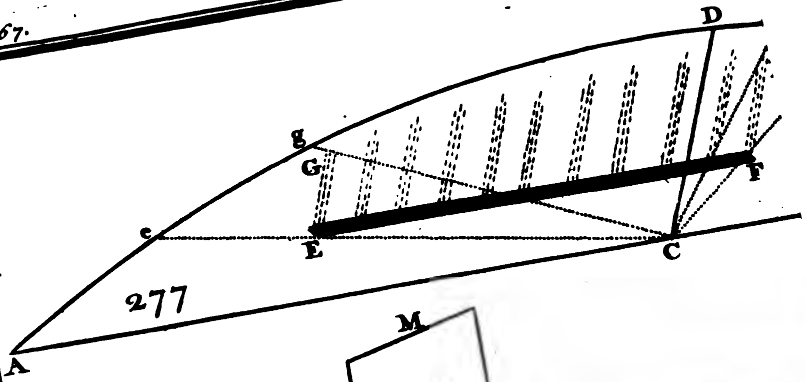
[1] Phil. Tranſ. N°. 369. p. 214. l. 6.

[2] Ibid. p. 211. l. 23.

[3] 21. Dec. 1729. at 7. p. m.



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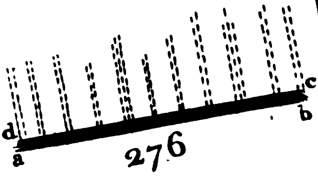

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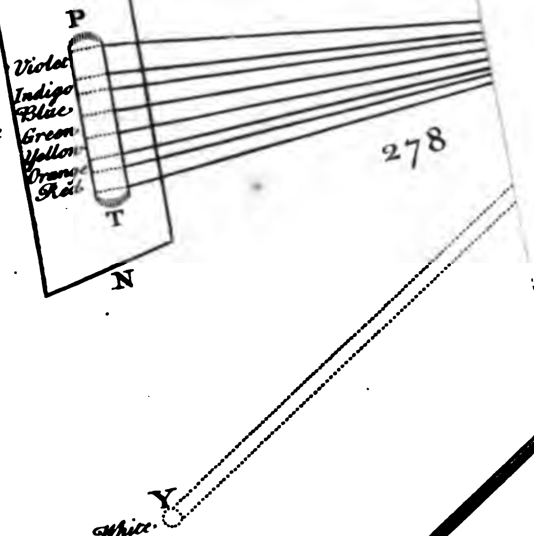

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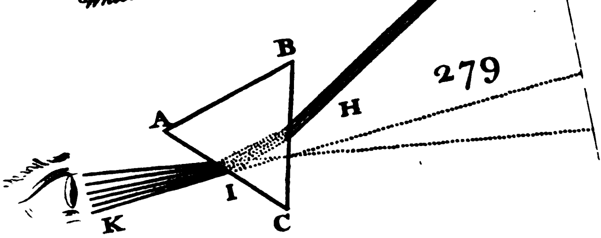

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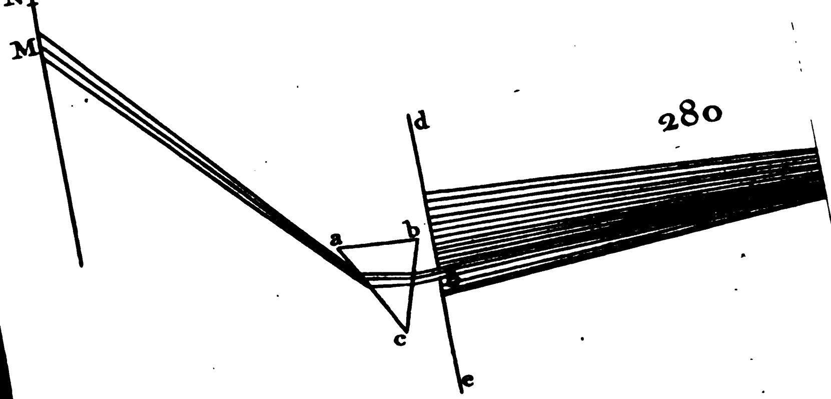

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tal diameter *cd*, drawn through the moon, as 4 to 3, pretty nearly. The oval in the figure is drawn according to theſe proportions to be compared with the appearance of a halo when it ſhall happen.

Figure of the ſun and moon conſidered.

168. What has been ſaid of the oval projection of a halo, is applicable to the ſun or moon, whoſe projections are likewiſe oval, eſpecially near the horizon; but whether they appear ſo or not, is hard to judge, becauſe this oval projection is ſo very ſmall and ſo remote, that we cannot well perceive a ſenſible difference in the diſtances of its upper and under limbs from the eye, and conſequently can judge of its figure by no other perception but that of the figure of its picture upon the retina. On the contrary the ſun in the horizon has been often obſerved to appear oval in a contrary poſition[1], and by the preſent tables of refractions in the air, the angle ſubtended by its horizontal and vertical diameters are to each other in the ratio of 5 to 4 reſpectively, or thereabouts; becauſe the loweſt ray is more refracted than the higheſt. By this means the picture of the ſun upon the retina becomes oval, and cauſes this oval appearance.

And confirmed by the appearance of comets tails and meteors.

169. The preſent theory is alſo confirmed by the appearances of the tails of comets; which, whatever be their real figure, magnitude and ſituation in abſolute ſpace, do always appear to be an arch of the concave ſky. Upon the whole it appears to me that the judgments we make of the apparent place, magnitude, ſhape, and poſition of all remote objects in the heavens, as of the ſun, moon, comets, conſtellations, rainbows, halos and all other meteors, are the very ſame as they would be if we viewed their perſpective draughts traced out by the viſual rays upon a real ſurface in the place and figure of the apparent concavity of the ſky. In confirmation of which, I will conclude with Mr. *Cotes*'s account of a remarkable meteor (ſeen on the 6th of March 17$\frac{15}{16}$) written in anſwer to a letter which he received from Dr. *Dannye* then Rector of *Spofferth* in Yorkſhire.

The optical appearance of a remarkable meteor, ſolved by Mr. *Cotes*.

170. The appearance of the meteor was very nearly the ſame with us here at *Cambridge* as with you, excepting that the triangular ſtreams of light were not ſo permanent as you ſeem to deſcribe them, and the point to which they all converged was diſtant from the zenith about 20 degrees, its azimuth lying between the ſouth and the eaſt at about 10 degrees from the ſouth, towards which point of the compaſs the wind tended. The poſition of this point of convergence may be more accurately determined, if there be occaſion: for at a quarter after ſeven when the appearance to us was in the greateſt perfection it lay nearly in the middle between the two bright ſtars in the heads of Caſtor and Pollux. I am told that ſome ſtreams were ſeen to ſhoot forth immediately after ſun ſet, and that they did not perfectly ceaſe till about 3 or 4 in the morning.

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[1] See *Scheiner*'s Refractiones cœleſtes ſive Solis Elliptici Phænomenon.

It was after ſeven before I had notice of this uncommon ſight. At firſt I ſaw only two or three of the triangular ſtreams towards the north and north weſt: theſe were not of long duration but were ſucceeded by others which appeared and vaniſhed again by turns, ariſing from and aſcending up to places in the heavens of very different altitudes above the horizon. From the time I began to view them they continued to aſcend more and more copiouſly, being propagated ſtill further and further from the north towards the weſt and eaſt and directed always to the heads of Gemini, till at length when they ſeemed almoſt to meet at the point of convergence, they began to aſcend up towards it from the ſouthern parts alſo and all around it; inſomuch that at a quarter after ſeven we had a perfect canopy of rays over us. The bottom of this canopy did no where reach down to the horizon; for near the north, where it deſcended the moſt, its altitude was about 10 or 15 degrees, and near the ſouth, where it deſcended the leaſt, its altitude was about 40 degrees. It remained in this ſtate about 2 minutes during which time we ſaw ſeveral colours ſome fainter and more permanent, others brighter, but quickly vaniſhing. Thus in the weſt I obſerved the rays to be tinged for ſome conſiderable time with an obſcure and heavy red: and in one of the brighteſt ſtreams at another time, there ſuddenly broke out a very vivid red which was inſtantly and gradually ſucceeded by the other priſmatick colours, all vaniſhing in about a ſecond of time. Theſe colours affected the ſenſe ſo ſtrongly that I thought them to be more intenſe than thoſe of the brighteſt rain-bow I had ever ſeen. A ſmall time before the appearance loſt its perfection, we were ſurprized to obſerve a ſhaking and trembling of the ſtreams chiefly in their upper parts, during which their convergence was confounded, and the whole heavens ſeemed to be in a convulſion. At the ſame time I could perceive waves of light towards the north which moved upwards and in their motion croſſed the ſtreams lying parallel to the horizon. Theſe waves were different from thoſe broad ones which you mention, and which I alſo took notice of: their breadth ſeemed to be about a degree, their length about 90 degrees; and I can compare them to nothing better than to thoſe ſlender waves upon the ſurface of ſtagnant water which are made by caſting in a ſmall ſtone.

Fig. 276. About ſeven or eight years ago, I happened to ſee a meteor which it will be of uſe to deſcribe to you. Along the horizon in the north there lay a white and luminous, and ſeemingly denſe matter in the form of a cloud repreſented by *abcd*; the length of it, *ab*, was about 10 or 15 degrees. From this there aroſe directly upwards pointed ſtreams of the like luminous and white matter which yet did not appear in any part of it to be ſo denſe as the former; and grew gradually more and more rare in its upper parts ſo as to vaniſh almoſt inſenſibly at the points. There was ſome little difference in the height of theſe ſtreams but they generally

aſcended up to about 4 degrees above the horizon. They were very numerous and contiguous to each other, and ſeemed to be compoſed of very ſlender parallel filaments or rays. This was the common appearance, and the only remarkable thing which I farther obſerved was, that ſometimes a fire or flame would break out in the cloud, *abcd*, and move along it in a direction parallel to the horizon: and during this motion a pointed ſtream directly over the fire ſeemed to run along with it, and to paſs by the other more fixed ſtreams to which it always kept it ſelf parallel.

I am perſwaded that the late appearance was of the ſame kind with this which I have now been deſcribing. For let *AB* repreſent the plane of the horizon, *C* the place of the ſpectator, *EF* a fund of vapours or exhalations at a conſiderable height above us, diffuſed every way into a large and ſpacious plane parallel to the horizon. This fund of mixt matter by fermentation will emit ſtreams from it ſelf ſuch as *EG*, *FH*, &c. which if the wind be perfectly ſtill will aſcend perpendicularly upwards; if it be boiſterous and irregular they will be blended and confounded together, but if it be very gentle and uniform as it was at the time of our appearance they will be inclined towards the point of the horizon which is oppoſite to that from which the wind blows. Now if *ADB* repreſents the concave of the heavens, and a line *CD*, be drawn pararallel to the columns *EG*, *FH*, &c. it is certain by the rules of perſpective, that theſe columns will appear upon that concave to converge all around towards the point *D*: thus the column, *EG*, will ſeem to ariſe from the point *e*, to aſcend up to *g*, and to take up the ſpace *eg*: and in like manner the arch *fh*, will be the projection of the column *FH*. From hence it is evident that the reaſon why the triangular ſtreams aſcended at firſt only from the northern parts of the heavens was this: the fund of matter, *EF*, was not yet arrived by its motion to the line *CD*. After it had paſſed that line it is plain they muſt appear to aſcend from all quarters. A great number of columns being therefore diſpoſed to emit light at the ſame time cauſed that perfect canopy, which I deſcribed above. The reaſon why that canopy deſcended lower in the north, than in the ſouth was this: the ſhining columns which had not yet paſſed the line *CD* were more numerous and more remote from it than thoſe which had paſſed it, for if the point *E*, be farther diſtant from *CD* than the point *F*, the arch *Ae*, muſt needs be leſs than the arch *Bf*. An irregular guſt of wind blowing upon and ſhaking the columns (I ſuppoſe) was the cauſe of that trembling which appeared in the triangular ſtreams, and the cauſe alſo which deſtroyed that fine appearance of the canopy. The ſlender circular waves ſeen at the ſame time, might alſo be explained from the ſame cauſe. I need not detain you any longer, by endeavouring to make out ſome other particulars of this unuſual appearance: I fear I have been already Fig. 277.

ready too tedious. However, I will not omit to mention a very eaſy contrivance by which the thing may be tolerably well repreſented to view. Take a hoop, and round about it faſten ſeveral ſtraight ſticks parallel to each other but all inclined to the plane of the hoop, hold this plane parallel to the horizon, and in that poſture move it with its ſticks over a candle, the ſhadow of the ſticks upon the ceiling of your room, will converge to a point, not directly over the candle (as they would have done had the ſticks been perpendicular to the plane of the hoop) but to the point in which a line drawn from the candle parallel to the ſticks, ſhall interſect the plane of the ceiling.

CHAP. VI.

CONCERNING THE ORIGINE AND CAUSE OF COLOURS.

Deſign.

To make this popular treatiſe more compleat, I have added Sir *Iſaac Newton*'s theory of colours, deſcribed in his own words as near as poſſible and proved by his own experiments.

I. Experiment. A deſcription of the ſun's image made by a priſm. Newt. Opt. p. 21. Fig. 278.

171. In a very dark chamber at a round hole *F*, about one third of an inch broad, made in the ſhut of a window, I placed a glaſs priſm *ABC* whereby the beam of the ſun's light *SF*, which came in at that hole, might be refracted upwards, toward the oppoſite wall of the chamber, and there form a coloured image of the ſun, repreſented at *PT*. The axis of the priſm, (that is the line paſſing through the middle of the priſm, from one end of it to the other end, parallel to the edge of the refracting angle) was in this and the following experiments perpendicular to the incident rays. About this axis I turned the priſm ſlowly, and ſaw the refracted light on the wall, or coloured image of the ſun, firſt to deſcend, and then to aſcend. Between the deſcent and aſcent when the image ſeemed ſtationary, I ſtopped the priſm and fixt it in that poſture.

Then I let the refracted light fall perpendicularly upon a ſheet of white paper, *MN*, placed at the oppoſite wall of the chamber, and obſerved the figure and dimenſions of the ſolar image, *PT*, formed on the paper by that light. This image was oblong and not oval, but terminated by two rectilinear and parallel ſides and two ſemicircular ends. On its ſides it was bounded pretty diſtinctly, but on its ends very confuſedly and indiſtinctly, the light there decaying and vaniſhing by degrees. At the diſtance of $18\frac{1}{2}$ feet from the priſm the breadth of the image was about $2\frac{1}{8}$ inches, but its length was about $10\frac{1}{4}$ inches, and the length of its rectilinear ſides about 8 inches; and *ACB* the refracting angle of the priſm, whereby ſo great a length was made, was 64 degrees. With a leſs angle the length of the image was leſs, the breadth remaining the ſame. It is farther to be obſerved that the rays went on in ſtraight lines from the priſm to the image, and therefore at their going out of the priſm had all that inclination to one another from which the length of the image proceeded. This image PT was coloured, and the more eminent colours lay in this order from the bottom at T to the top at P; red, orange, yellow, green, blue, indigo, violet; together with all their intermediate degrees in a continual ſucceſſion perpetually varying.

Hence the ſun's rays are differently refrangible.

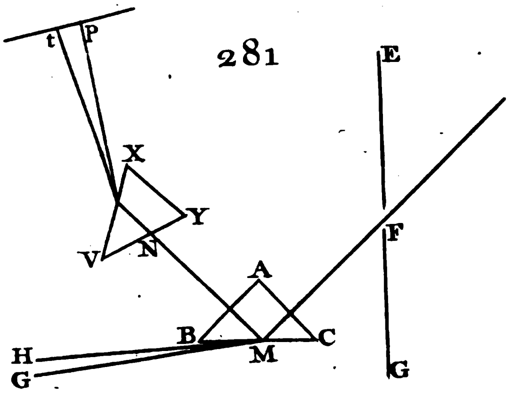
172. Our author concludes from this experiment, and many more to be mentioned hereafter, that the light of the ſun conſiſts of a mixture of ſeveral ſorts of coloured rays, ſome of which at equal incidences are more refracted than others, and therefore are called more refrangible. The red at T, being neareſt to the place Y, where the rays of the ſun would go directly if the priſm was taken away, is the leaſt refracted of all the rays; and the orange, yellow, green, blue, indigo and violet are continually more and more refracted, as they are more and more diverted from the courſe of the direct light. For by mathematical reaſoning he has proved, that when the priſm is fixt in the poſture above mentioned, ſo that the place of the image ſhall be the loweſt poſſible, or at the limit between its deſcent and aſcent, the figure of the image ought then to be round like the ſpot at Y, if all the rays that tended to it were equally refracted. Therefore ſeeing by experience it is found that this image is not round, but about 5 times longer than broad, it follows that all the rays are not equally refracted. And this concluſion is farther confirmed by the following experiments.

II. Experiment. Newt. Opt. p. 27. Fig. 279.

In the ſun beam SF which was propagated into the room through the hole in the window-ſhut EG, at the diſtance of ſome feet from the hole, I held the priſm ABC in ſuch a poſture, that its axis might be perpendicular to that beam: then I looked through the priſm upon the hole F, and turning the priſm to and fro about its axis to make the image pt, of the hole aſcend and deſcend, when between its two contrary motions it ſeemed ſtationary, I ſtopped the priſm; in this ſituation of the priſm viewing through it the ſaid hole F, I obſerved the length of its refracted image pt to be many times greater than its breadth; and that the moſt refracted part thereof appeared violet at p; the leaſt refracted red, at t; and the middle parts indigo, blue, green, yellow and orange in order. The ſame thing happened when I removed the priſm out of the ſun's light and looked through it upon the hole ſhining by the light of the clouds beyond it. And yet if the refractions of all the rays were equal according to one certain proportion of the ſines of incidence and refraction, as is vulgarly ſuppoſed, the refracted image ought to have appeared round, by the mathematical demonſtration above mentioned. So then by theſe two experiments it appears that in equal incidences there is a conſiderable inequality of refractions.

For the diſcovery of this fundamental property of light, which has opened the whole myſtery of colours, we ſee our author was not only beholden to the experiments themſelves, which many others had made before him, but alſo to his ſkill in geometry; which was abſolutely neceſſary to determine what the figure of the refracted image ought to be upon the old principle of an equal refraction of all the rays: but having thus made the diſcovery he contrived the following experiment to prove it at ſight.

III. Experiment. Newt. Opt. p. 37. Fig. 280. In the middle of two thin boards, *DE*, *de*, I made a round hole in each, at *G* and *g*, a third part of an inch in diameter; and in the window-ſhut a much larger hole being made, at *F*, to let into my darkened chamber a large beam of the ſun's light, I placed a priſm, *ABC*, behind the ſhut in that beam, to refract it towards the oppoſite wall; and cloſe behind this priſm I fixed one of the boards *DE*, in ſuch manner that the middle of the refracted light might paſs through the hole made in it at *G*, and the reſt be intercepted by the board. Then at the diſtance of about 12 feet from the firſt board I fixed the other board *de*, in ſuch manner that the middle of the refracted light, which came through the hole in the firſt board, and fell upon the oppoſite wall, might paſs through the hole *g* in this other board *de*, and the reſt being intercepted by the board might paint upon it the coloured ſpectrum of the ſun. And cloſe behind this board I fixed another priſm *abc* to refract the light which came through the hole *g*. Then I returned ſpeedily to the firſt priſm *ABC* and by turning it ſlowly to and fro about its axis, I cauſed the image which fell upon the ſecond board *de* to move up and down upon that board, that all its parts might paſs ſucceſſively through the hole in that board, and fall upon the priſm behind it. And in the mean time I noted the places, *M*, *N*, on the oppoſite wall, to which that light after its refraction in the ſecond priſm did paſs; and by the difference of the places at *M* and *N*, I found that the light, which being moſt refracted in the firſt priſm *ABC*, did go to the blue end of the image, was again more refracted by the ſecond priſm *abc*, than the light which went to the red end of that image. For when the lower part of the light which fell upon the ſecond board *de*, was caſt through the hole *g*, it went to a lower place *M* on the wall; and when the higher part of that light was caſt through the ſame hole *g*, it went to a higher place *N* on the wall; and when any intermediate part of the light was caſt through that hole, it went to ſome place in the wall between *M* and *N*. The unchanged poſition of the holes in the boards made the incidence of the rays upon the ſecond priſm to be the ſame in all caſes. And yet in that common incidence ſome of the rays were more refracted and others leſs: and thoſe were more refracted in this priſm, which by a greater refraction in the firſt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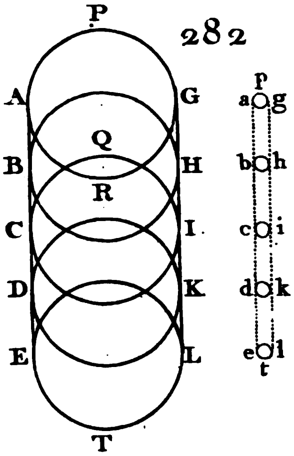
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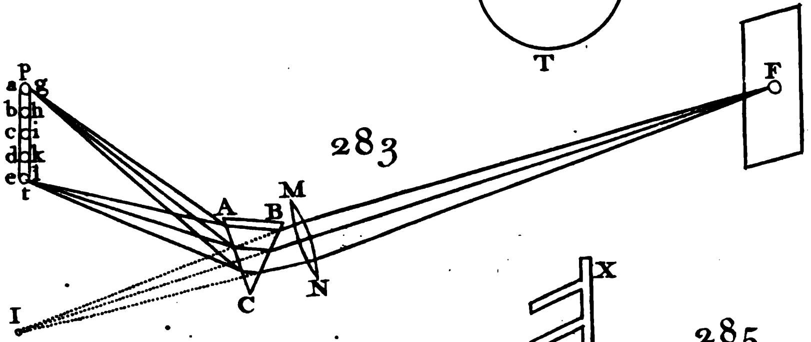
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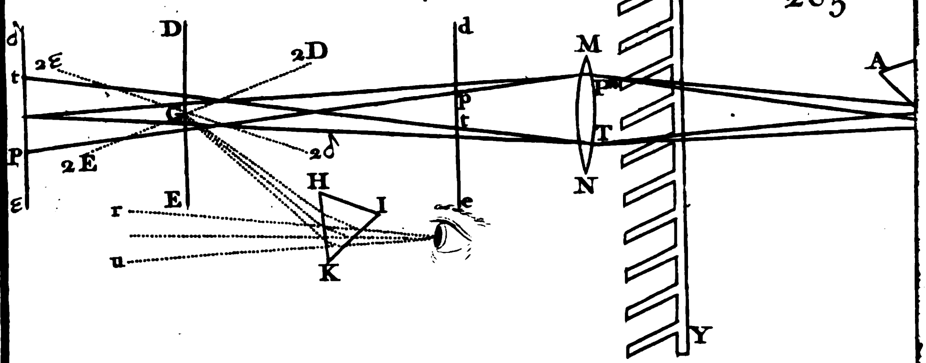
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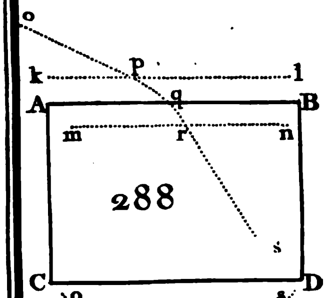
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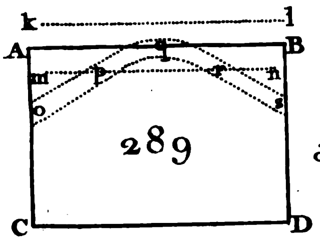
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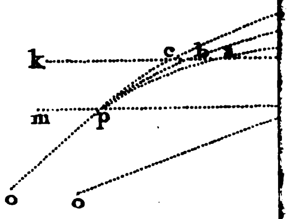

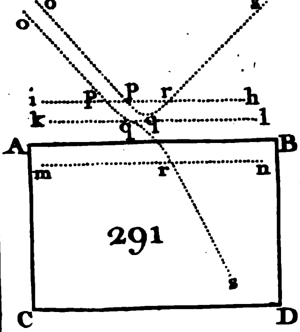
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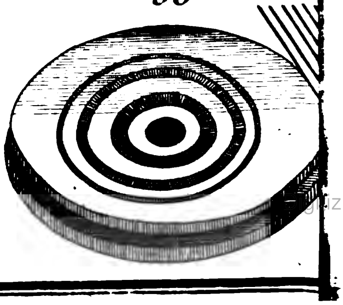

firſt priſm were more turned out of their way; and therefore for their conſtancy of being more refracted are deſervedly called more refrangible.

Our author ſhews alſo, by experiments made with a convex glaſs, that lights (reflected from natural bodies) which differ in colour, differ alſo in degrees of refrangibility [a]: and that they differ in the ſame manner as the rays of the ſun do.

IV. Experiment. Rays of the ſun are differently reflexible. Newt. Opt. p. 45. Fig. 281.

173. The ſun's light conſiſts of rays differing in reflexibility, and thoſe rays are more reflexible than others which are more refrangible. A priſm *ABC* whoſe two angles at its baſe *BC* were equal to one another and half right ones, and the third at *A* a right one, I placed in a beam *FM* of the ſun's light, let into a dark chamber through a hole *F* one third part of an inch broad. And turning the priſm ſlowly about its axis until the light which went through one of its angles *ACB* and was refracted by it to *G* and *H*, began to be reflected into the line *MN* by its baſe *BC*, at which till then it went out of the glaſs; I obſerved that thoſe rays as *MH* which had ſuffered the greateſt refraction were ſooner reflected than the reſt. To make it evident that the rays which vaniſhed at *H* were reflected into the beam *MN*, I made this beam paſs through another priſm *VXY*, and being refracted by it to fall afterwards upon a ſheet of white paper *pt* placed at ſome diſtance behind it and there by that refraction to paint the uſual colours at *pt*. Then cauſing the firſt priſm to be turned about its axis according to the order of the letters *ABC*, I obſerved that when thoſe rays *MH* which in this priſm had ſuffered the greateſt refraction and appeared blue and violet, began to be totally reflected, the blue and violet light on the paper which was moſt refracted in the ſecond priſm received a ſenſible increaſe at *p*, above that of the red and yellow at *t*: and afterwards when the reſt of the light, which was green yellow and red began to be totally reflected and vaniſhed at *G*, the light of thoſe colours at *t* on the paper *pt* received as great an increaſe as the violet and blue had received before. Which puts it paſt diſpute that thoſe rays became firſt of all totally reflected at the baſe *BC*, which before at equal incidences with the reſt upon the baſe *BC* had ſuffered the greateſt refraction. I do not here take notice of any refractions made in the ſides *AC*, *AB* of the firſt priſm, becauſe the light enters almoſt perpendicularly at the firſt ſide and goes out almoſt perpendicularly at the ſecond, and therefore ſuffers none, or ſo little that the angles of incidence at the baſe *BC* are not ſenſibly altered by it; eſpecially if the angles of the priſm at the baſe *BC* be each about 40 degrees. For the rays *FM* begin to be totally reflected when the angle *CMF* is about 50 degrees [b], and therefore they will then make a right angle of 90 degrees with *AC*.

It appears alſo from this experiment that the beam of light *MN*, reflected by the baſe of the priſm, being augmented firſt by the more re-

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[a] Newt. Opt. p. 16.

[b] Art. 17.

frangible rays and afterwards by the leſs refrangible is compoſed of rays differently refrangible.

Definitions. Newt. Opt. p. 4.

The light whoſe rays are all alike refrangible I call ſimple homogeneal and ſimilar, and that whoſe rays are ſome more refrangible than others I call compound, heterogeneal and diſſimilar. The former light I call homogeneal not becauſe I would affirm it ſo in all reſpects; but becauſe the rays which agree in refrangibility agree at leaſt in all their other properties which I conſider in the following diſcourſe.

The colours of homogeneal lights I call primary, homogeneal and ſimple, and thoſe of heterogeneal lights, heterogeneal and compound. For theſe are always compounded of homogeneal lights, as will appear in the following diſcourſe.

Newt. Opt. p. 108.

The homogeneal light and rays which appear red, or rather make objects appear ſo, I call rubrifick or red-making; thoſe which make objects appear yellow, green, blue and violet, I call yellow-making, green-making, blue-making, violet-making; and ſo the reſt. And if at any time I ſpeak of light and rays as coloured or endued with colours, I would be underſtood to ſpeak not philoſophically and properly but groſly, and according to ſuch conceptions as vulgar people in ſeeing all theſe experiments would be apt to frame. For the rays to ſpeak properly are not coloured. In them there is nothing elſe than a certain power and diſpoſition to ſtir up a ſenſation of this or that colour. For as ſound in a bell or muſical ſtring or other ſounding body, is nothing but a trembling motion, and in the air nothing but that motion propagated from the object, and in the ſenſorium it is a ſenſe of that motion under the form of ſound; ſo colours in the object are nothing but a diſpoſition to reflect this or that ſort of rays more copiouſly than the reſt; in the rays they are nothing but their diſpoſitions to propagate this or that motion into the ſenſorium; and in the ſenſorium they are ſenſations of thoſe motions under the forms of colours.

How the long image is compoſed of circles of different ſorts of rays. Opt. p. 31. Fig. 282.

174. By the mathematical propoſition above mentioned[a], it is certain that the rays which are equally refrangible do fall upon a circle anſwering to the ſun's apparent diſque, which will alſo be proved by experiment by and by. Now let *AG* repreſent the circle which all the moſt refrangible rays, propagated from the whole diſque of the ſun, would illuminate and paint upon the oppoſite wall if they were alone; *EL* the circle which all the leaſt refrangible rays would in like manner illuminate if they were alone; *BH*, *CI*, *DK* the circles which ſo many intermediate ſorts would paint upon the wall, if they were ſingly propagated from the ſun in ſucceſſive order, the reſt being intercepted; and conceive that there are other circles without number, which innumerable other intermediate ſorts of rays would ſucceſſively paint upon the wall, if the ſun ſhould ſucceſſively emit every ſort apart. And ſeeing the ſun emits

[a] Art. 172.

emits all theſe ſorts at once, they muſt all together illuminate and paint innumerable equal circles; of all which, being according to their degrees of refrangibility placed in order in a continual ſeries, that oblong ſpectrum *PT* is compoſed, which was deſcribed in the firſt experiment.

How thoſe ſorts of rays may be farther ſeparated. Newt. Opt. p. 54. Fig. 282.

175. Now if theſe circles whilſt their centers keep their diſtances and poſitions could be made leſs in diameter, their interfering one with another and conſequently the mixture of the heterogeneous rays would be proportionably diminiſhed. Let the circles *AG*, *BH*, *CI*, &c. remain as before; and let *ag*, *bh*, *ci*, &c. be ſo many leſs circles lying in a like continual ſeries, between two parallel right lines *ae* and *gl*, with the ſame diſtances between their centers, and illuminated with the ſame ſorts of rays: that is, the circle *ag* with the ſame ſort by which the correſponding circle *AG* was illuminated; and the reſt of the circles *bh*, *ci*, *dk*, *el* reſpectively with the ſame ſorts of rays by which the correſponding circles *BH*, *CI*, *DK*, *EL*, were illuminated. In the figure *PT* compoſed of the great circles, three of thoſe circles *AG*, *BH*, *CI* are ſo expanded into each other, that three ſorts of rays, by which thoſe circles are illuminated, together with innumerable other ſorts of intermediate rays, are mixed at *QR* in the middle of the circle *BH*. And the like mixture happens throughout almoſt the whole length of the figure *PT*. But in the figure *pt*, compoſed of the leſs circles, the three leſs circles *ag*, *bh*, *ci*, which anſwer to thoſe three greater do not extend into one another; nor are there any where mingled ſo much as any two of the three ſorts of rays by which thoſe circles are illuminated, and which in the figure *PT* are all of them intermingled at *QR*. So then if we would diminiſh the mixture of the rays we are to diminiſh the diameters of the circles. Now theſe would be diminiſhed if the ſun's diameter, to which they anſwer, could be made leſs than it is, or (which comes to the ſame purpoſe) if without doors, at a great diſtance from the priſm towards the ſun, ſome opake body were placed with a round hole in the middle of it to intercept all the ſun's light, except ſo much as coming from the middle of his body could paſs through that hole to the priſm. For ſo the circles *AG*, *BH* and the reſt, would not any longer anſwer to the whole diſque of the ſun, but only to that part of it which could be ſeen from the priſm through that hole; that is to the apparent magnitude of that hole viewed from the priſm. But that theſe circles may anſwer more diſtinctly to that hole, a lens is to be placed by the priſm to caſt the image of the hole, (that is every one of the circles *AG*, *BH*, &c.) diſtinctly upon the paper at *PT*; after ſuch a manner as by a lens placed at a window the pictures of objects abroad are caſt diſtinctly upon a paper within the room. If this be done it will not be neceſſary to place that hole very far off, no not beyond the window. And therefore inſtead of that hole I uſed the hole in the window-ſhut as follows.

V. Experiment. Newt. Opt. p. 57. Fig. 283.

In the ſun's light let into my darkned chamber through a ſmall round hole in my window-ſhut, at about ten or twelve feet from the window, I placed a lens *MN*, by which the image of the hole *F*, might be diſtinctly caſt upon a ſheet of white paper placed at *I*. Then immediately after the lens I placed a priſm *ABC*, by which the trajected light might be refracted either upwards or ſideways, and thereby the round image which the lens alone did caſt upon the paper at *I*, might be drawn out into a long one with parallel ſides as repreſented at *pt*. This oblong image I let fall upon another paper at about the ſame diſtance from the priſm as the image at *I*, moving the paper either towards the priſm or from it, untill I found the juſt diſtance where the rectilinear ſides of the image *pt* became moſt diſtinct. For in this caſe the circular images of the hole, which compoſe that image, after the manner that the circles *ag*, *bh*, *ci*, &c. do the figure *pt*, were terminated moſt diſtinctly, and therefore extended into one another the leaſt that they could, and by conſequence the mixture of the heterogeneous rays was now the leaſt of all. The circles, *ag*, *bh*, *ci*, &c, which compoſe the image *pt*, are each equal to the circle at *I*; and therefore, by diminiſhing the hole *F* or by removing the lens farther from it, may be diminiſhed at pleaſure, whilſt their centers keep the ſame diſtances from each other. Thus by diminiſhing the breadth of the image *pt* the circles of heterogeneal rays that compoſe it, may be ſeparated from each other as much as you pleaſe. Yet inſtead of the circular hole *F*, it is better to ſubſtitute an oblong hole ſhaped like a parallelogram with its length parallel to the length of the priſm. For if this hole be an inch or two long and but a tenth or twentieth part of an inch broad, or narrower, the light of the image *pt* will be as ſimple as before or ſimpler; and the image being much broader is therefore fitter to have experiments tried in its light than before.

VI. Experiment. Homogeneal light is refracted regularly &c. Newt. Opt. p. 62.

176. Homogeneal light is refracted regularly without any dilatation ſplitting or ſhattering of the rays, and the confuſed viſion of objects ſeen through refracting bodies by heterogeneal light, ariſes from the different refrangibility of ſeveral ſorts of rays. This will appear by the experiments which follow. In the middle of a black paper I made a round hole about a fifth or a ſixth part of an inch in diameter. Upon this paper I cauſed the ſpectrum of homogeneal light deſcribed in the former article, ſo to fall that ſome part of the light might paſs through the hole in the paper. This tranſmitted part of the light I refracted with a priſm placed behind the paper; and letting this refracted light fall perpendicularly upon a white paper two or three feet diſtant from the priſm, I found that the ſpectrum formed on the paper by this light was not oblong, as when it is made, in the firſt experiment, by refracting the ſun's compound light, but was (ſo far as I could judge by my eye) perfectly circular; the length being no where greater than the breadth; which ſhews that this

light is refracted regularly without any dilatation of the rays; and is an ocular demonſtration of the mathematical propoſition mentioned in the 172d article.

VII. Experiment. Newt. Opt. p. 63. In the homogeneal light I placed a paper circle of a quarter of an inch in diameter; and in the ſun's unrefracted, heterogeneal, white light I placed another paper circle of the ſame bigneſs; and going from theſe papers to the diſtance of ſome feet I viewed both circles through a priſm. The circle illuminated by the ſun's heterogeneal light appeared very oblong as in the 2d experiment, the length being many times greater than the breadth. But the other circle illuminated with homogeneal light appeared circular and diſtinctly defined, as when it is viewed by the naked eye; which proves the whole propoſition mentioned at the beginning of this article.

VIII. Experiment. Ibid. In the homogeneal light I placed flies and ſuch like minute objects, and viewing them through a priſm I ſaw their parts as diſtinctly defined as if I had viewed them with the naked eye. The ſame objects placed in the ſun's unrefracted heterogeneal light which was white, I viewed alſo through a priſm, and ſaw them moſt confuſedly defined, ſo that I could not diſtinguiſh their ſmaller parts from one another. I placed alſo the letters of a ſmall print one while in the homogeneal light and then in the heterogeneal, and viewing them through a priſm they appeared in the latter caſe ſo confuſed and indiſtinct that I could not read them; but in the former they appeared ſo diſtinct that I could read readily, and thought I ſaw them as diſtinct as when I viewed them with my naked eye; in both caſes I viewed the ſame objects through the ſame priſm at the ſame diſtance from me and in the ſame ſituation. There was no difference but in the lights by which the objects were illuminated and which in one caſe was ſimple in the other compound; and therefore the diſtinct viſion in the former caſe and confuſed in the latter could ariſe from nothing elſe than from that difference in the lights. Which proves the whole propoſition.

The colour of homogeneal light cannot be changed by refractions nor by reflexions. Newt. Opt. p. 107.

177. In theſe three experiments it is farther very remarkable that the colour of homogeneal light was never changed by the refraction: and as theſe colours were not changed by refractions; ſo neither were they by reflexions. For all white, grey, red, yellow, green, blue, violet bodies, as paper, aſhes, red lead, orpiment, indigo, biſe, gold, ſilver, copper, graſs, blue flowers, violets, bubbles of water tinged with various colours, peacocks feathers, the tincture of *lignum nephriticum* and ſuch like, in red homogeneal light appeared totally red, in blue light totally blue, in green light totally green, and ſo of other colours. In the homogeneal light of any colour they all appeared totally of that ſame colour, with this only difference, that ſome of them reflected that light more ſtrongly,

others more faintly. I never yet found any body which by reflecting homogeneal light could ſenſibly change its colour.

From all which it is manifeſt, that if the ſun's light conſiſted of but one ſort of rays, there would be but one colour in the world. Nor would it be poſſible to produce any new colour by reflexions and refractions: and by conſequence that the variety of colours depends upon the compoſition of light.

All homog. light has its proper colour anſwering to its degree of refrangibility. Newt. Opt. p 106.

178. The ſolar image *pt* formed by the ſeparated rays in the 5th experiment, did in the progreſs from its end *p*, on which the moſt refrangible rays fell, unto its end *t*, on which the leaſt refrangible rays fell, appear tinged with this ſeries of colours; violet, indigo, blue, green, yellow, orange, red, together with all their intermediate degrees in a continual ſucceſſion perpetually varying: ſo that there appeared as many degrees of colours as there were ſorts of rays differing in refrangibility. And ſince theſe colours could not be changed by refractions nor by reflexions, it follows that all homogeneal light has its proper colour anſwering to its degree of refrangibility.

The ſine of incidence of every homog. ray is to its ſine of refraction in a given ratio. Newt. Opt. p. 64. Fig. 284.

179. Every homogeneal ray conſidered apart is refracted according to one and the ſame rule[a], ſo that its ſine of incidence is to its ſine of refraction in a given ratio: that is, every different coloured ray has a different ratio belonging to it. This our author has proved by experiment, and by other experiments has determined by what numbers thoſe given ratios are expreſſed. For inſtance, if an heterogeneal white ray of the ſun emerges out of glaſs into air, or which is the ſame thing, if rays of all colours be ſuppoſed to ſucceed one another in the ſame line *AC*, and *AD* their common ſine of incidence in glaſs be divided into 50 equal parts, then *EF* and *GH* the ſines of refraction into air, of the leaſt and moſt refrangible rays will be 77 and 78 ſuch parts reſpectively. And ſince every colour has ſeveral degrees, the ſines of refraction of all the degrees of red will have all intermediate degrees of magnitude from 77 to $77\frac{1}{8}$, of all the degrees of orange from $77\frac{1}{8}$ to $77\frac{1}{5}$, of yellow from $77\frac{1}{5}$ to $77\frac{1}{3}$, of green from $77\frac{1}{3}$ to $77\frac{1}{2}$, of blue from $77\frac{1}{2}$ to $77\frac{2}{3}$, of indigo from $77\frac{2}{3}$ to $77\frac{7}{9}$, and of violet from $77\frac{7}{9}$ to 78[b].

The different properties of ſimple and compound colours. Newt. Opt. p. 115.

180. Colours may be produced by compoſition which ſhall be like to the colours of homogeneal light, as to the appearance of colour, but not as to the immutability of colour and conſtitution of light. And thoſe colours, by how much they are more compounded, by ſo much are they leſs full and intenſe; and by too much compoſition they may be diluted and weakened till they ceaſe, and the mixture becomes white or grey. There may be alſo colours produced by compoſition, which are not fully like any of the colours of homogeneal light. For a mixture of homogeneal red and yellow compounds an orange, like in appearance of colour to that orange which in the ſeries of unmixed priſmatick colours lyes between

[a] Art. 13.

[b] Newt. Opt. p. 109.

between them. But the light of one orange is homogeneal as to refrangibility, that of the other is heterogeneal; and the colour of the one, if viewed through a priſm remains unchanged, that of the other is changed and reſolved into its component colours red and yellow. And after the ſame manner other neighbouring homogeneal colours may compound new colours, like the intermediate homogeneal ones: as yellow and green the colour between them both; and afterwards if blue be added there will be made a green, the middle colour of the three which enter the compoſition. For the yellow and blue on either hand, if they are equal in quantity, draw the intermediate green equally towards themſelves, and ſo keep it as it were in æquilibrio, that it verge not more to the yellow on one hand, than to the blue on the other, but by their mixed actions remain ſtill a middle colour. To this mixed green there may be farther added ſome red and violet, and yet the green will not preſently ceaſe but only grow leſs full and vivid; and by increaſing the red and violet, it will grow more and more dilute, untill by the prevalence of the added colours it be overcome and turned into whiteneſs or ſome other colour. So if to the colour of any homogeneal light, the ſun's white light compoſed of all ſorts of rays be added, that colour will not vaniſh or change its ſpecies, but be diluted, and by adding more and more white it will be diluted more and more perpetually. Laſtly if red and violet be mingled there will be generated according to their various proportions various purples: ſuch as are not like in appearance to the colour of any homogeneal light; and of theſe purples mixed with yellow and blue may be made other new colours.

181. Whiteneſs and all grey colours between white and black, may be compounded of colours; and the whiteneſs of the ſun's light is compounded of all the primary colours mixed in a due proportion.

IX. Experiment. Whiteneſs may be compounded of colours. Newt. Opt. p. 117. Fig. 285.

For let the ſolar image *PT* fall upon a lens *MN* above four inches broad and about ſix feet diſtant from the priſm *ABC*, and ſo figured that it may cauſe the coloured light which divergeth from the priſm to converge and meet again at its focus *G* about 6 or 8 feet diſtant from the lens, and there to fall perpendicular upon a white paper *DE*. And if you move this paper to and fro, you will perceive that near the lens, as at *de*, the whole ſolar image, ſuppoſe at *pt*, will appear upon it intenſely coloured after the manner above explained: and that by receding from the lens thoſe colours will perpetually come towards one another, and by mixing more and more dilute one another continually, untill at length the paper comes to the focus *G*, where by a perfect mixture they will wholly vaniſh and be converted into whiteneſs, the whole light appearing now upon the paper like a little white circle. And afterwards by receding farther from the lens, the rays which before converged will now croſs one another in the focus *G*, and diverge from thence and thereby make the colours to appear again, but yet in a contrary order, ſuppoſe at *δε*, where the red *t* is now above which before was below, and the violet *p* is below which before was above.

Let us now ſtop the paper at the focus *G* where the light appears totally white and circular, and let us conſider its whiteneſs. I ſay that this is compoſed of all the converging colours. For if any of thoſe colours be intercepted at the lens, the whiteneſs will ceaſe, and degenerate into that colour which ariſeth from the compoſition of the other colours which are not intercepted. And then if the intercepted colours be let paſs and fall upon that compound colour, they mix with it and by their mixture reſtore the whiteneſs. So if the violet, blue and green be intercepted, the remaining yellow, orange and red will compound upon the paper an orange, and then if the intercepted colours be let paſs, they will fall upon this compound orange and together with it decompound a white. So alſo if the red and violet be intercepted the remaining yellow, green and blue will compound a green upon the paper, and the red and violet being let paſs will fall upon this green, and together with it will decompound a white. And that in this compoſition of white, the ſeveral rays do not ſuffer any change in their colorifick qualities by acting upon one another, but are only mixed, and by a mixture of their colours produce white, may farther appear by theſe arguments.

If the paper be placed beyond the focus *G*, ſuppoſe at *δε*, and then the red colour at the lens be alternately intercepted and let paſs again, the violet colour on the paper will not ſuffer any change thereby, as it ought to do if the ſeveral ſorts of rays acted upon one another in the focus *G*, where they croſs. Neither will the red upon the paper be changed by any alternate ſtopping and letting paſs the violet which croſſes it.

And if the paper be placed at the focus *G* and the white round image at *G* be viewed through the priſm *HIK*, and by the refraction of that priſm be tranſlated to the place *rv* and there appear tinged with various colours; namely the violet at *v* and red at *r*, and others between; and then the red colour at the lens be often ſtopped and let paſs by turns, the red at *r* will accordingly diſappear and return as often; but the violet at *v* will not thereby ſuffer any change. And ſo by ſtopping and letting paſs alternately the blue at the lens, the blue at *v* will accordingly diſappear and return without any change made in the red at *r*. The red therefore depends on one ſort of rays and the blue on another ſort, which in the focus *G* where they were commixed do not act on one another. And there is the ſame reaſon of the other colours.

I conſidered farther that when the moſt refrangible rays *Pp* and the leaſt refrangible *Tt* are by converging inclined to one another, the paper if held very oblique to thoſe rays in the focus *G*, might reflect one ſort of them more copiouſly than the other ſort, and by that means the reflected ed light would be tinged in that focus with the colour of the predominant rays, provided thoſe rays ſeverally retained their colours or colorific qualities in the compoſition of white. But if they did not retain them in that white, but became all of them ſeverally endued there with a diſpoſition to ſtrike the ſenſe with the perception of white, then they could never loſe their whiteneſs by ſuch reflections. I inclined therefore the paper to the rays very obliquely, as repreſented in the poſture 2δ, 2ε, that the moſt refrangible rays *Pp*, falling more directly and conſequently more denſely upon it than the reſt, might be more copiouſly reflected than the reſt, and the whiteneſs at length changed ſucceſſively into blue, indigo and violet. Then I inclined it the contrary way as repreſented in the poſture 2*D*, 2*E*, that the leaſt refrangible rays *Tt* might fall more directly upon it, and conſequently be more copious in the reflected light than the reſt, and the whiteneſs turned ſucceſſively to yellow, orange and red.

Laſtly I made an inſtrument *XY* in faſhion of a comb whoſe teeth being in number 16 were about an inch and a half broad, and the intervals of the teeth about 2 inches wide. Then by interpoſing ſucceſſively the teeth of this inſtrument near the lens, I intercepted part of the colours by the interpoſed tooth, whilſt the reſt of them went on through the interval of the teeth to the paper *DE*, and there painted a round ſolar image. But the paper I had placed ſo that the image might appear white as often as the comb was taken away; and then the comb being interpoſed as was ſaid, that whiteneſs, by reaſon of the intercepted part of the colours at the lens, did always change into the colour compounded of thoſe colours which were not intercepted; and that colour was by the motion of the comb perpetually varied ſo, that in the paſſing of every tooth over the lens all theſe colours red, yellow, blue, green, purple did always ſucceed one another. I cauſed therefore all the teeth to paſs ſucceſſively over the lens, and when the motion was ſlow, there appeared a perpetual ſucceſſion of the colours upon the paper, but if I ſo accelerated the motion that the colours by reaſon of their quickneſs, could not be diſtinguiſhed from one another, the appearance of the ſingle colours ceaſed. There was no red, no yellow, no green, no blue, nor purple to be ſeen any longer, but from a confuſion of them all there aroſe one uniform white colour. Of the light which now by the mixture of all the colours appeared white, there was no part really white. One part was red another yellow, a third green, a fourth blue, a fifth purple, and every part retains its proper colour till it ſtrikes the ſenſorium. If the impreſſions follow one another ſlowly, ſo that they may be ſeverally perceived, there is made a diſtinct ſenſation of all the colours one after another in a continual ſucceſſion. But if the impreſſions follow one another ſo quick that they cannot be ſeverally perceived, there ariſeth out of them all one common ſenſation, which is neither of this colour alone, nor of that alone, but hath it ſelf indifferently to them all, and this is a ſenſation of whiteneſs. By the quickneſs of the ſucceſſions the impreſſions of the ſeveral colours are confounded in the ſenſorium, and out of that confuſion ariſeth a mixt ſenſation. If a burning coal be nimbly moved round in a circle with gyrations continually repeated, the whole circle will appear like fire. The reaſon of which is, that the ſenſation of the coal in the ſeveral places of that circle remains impreſſed on the ſenſorium till the coal returns again to the ſame place. And ſo in a quick conſecution of colours the impreſſion of every colour remains in the ſenſorium, until a revolution of all the colours be compleated and that firſt colour returns again. The impreſſions therefore of all the colours are at once in the ſenſorium and jointly ſtir up a ſenſation of them all: and ſo it is manifeſt from this experiment that the commixed impreſſions of all the colours do ſtir up and beget a ſenſation of white, that is, that whiteneſs is compounded of all the colours.

X. Experiment. Newt. Opt. p. 129.

Hitherto I have produced whiteneſs by mixing the priſmatick colours. If now the colours of natural bodies are to be mingled, let a little water thickened with ſoap be agitated to raiſe a froth, and after that froth has ſtood a little, there will appear to one that ſhall view it intently various colours every where in the ſurface of the ſeveral bubbles, but to one that ſhall go ſo far off that he cannot diſtinguiſh the colours from one another, the whole froth will grow white with a perfect whiteneſs.

XI. Experiment. Ibid.

Laſtly in attempting to compound a white by mixing the coloured powders which painters uſe, I conſidered that all coloured powders do ſuppreſs and ſtop in them a very conſiderable part of the light by which they are illuminated. For they become coloured by reflecting the light of their own colour more copiouſly, and that of all other colours more ſparingly; and yet they do not reflect the light of their own colours ſo copiouſly as white bodies do. If red lead, for inſtance, and a white paper be placed in the red light of the coloured ſpectrum made in the dark chamber by the refraction of a priſm, as is deſcribed in the 5th experiment, the paper will appear more lucid than the red lead, and therefore reflects the red-making rays more copiouſly than red lead doth. And if they be held in the light of any other colour, the light reflected by the paper will exceed the light reflected by the red lead in a much greater proportion. And the like happens in powders of other colours. And therefore by mixing ſuch powders we are not to expect a ſtrong and full white, ſuch as is that of paper, but ſome duſky obſcure one, ſuch as might ariſe from a mixture of light and darkneſs, or from white and black, that is a grey or dun, or ruſſet brown, ſuch as are the colours of a man's nails, of a mouſe, of aſhes, of ordinary ſtones, of mortar, of duſt and dirt in the high-ways, and the like; and ſuch a white I have often

ten produced by mixing coloured powders. For thus one part of red lead, and five parts of *viride æris* compoſed a dun colour like that of a mouſe. For theſe two colours were ſeverally ſo compounded of others, that in both together were a mixture of all colours; and there was leſs red lead than *viride æris* becauſe of the fulneſs of its colour. Again, one part of red lead and four parts of blue biſe, compoſed a dun colour verging a little to purple, and by adding to this a certain mixture of orpiment and *viride æris* in due proportion, the mixture loſt its purple tincture and became perfectly dun. But the experiment ſucceeded better without minium, thus. To orpiment I added by little and little a certain full bright purple, which painters uſe, until the orpiment ceaſed to be yellow, and became a pale red. Then I diluted that red by adding a little *viride æris* and a little more blue biſe than *viride æris*, until it became of ſuch a grey or pale white as verged to no one of the colours more than another: for thus it became of a colour equal in whiteneſs to that of aſhes or of wood newly cut or of a man's ſkin. The orpiment reflected more light than did any other of the powders, and therefore conduced more to the whiteneſs of the compound colour than they. To aſſign the proportions accurately may be difficult by reaſon of the different goodneſs of powders of the ſame kind. According as the colour of any powder is more or leſs full or luminous, it ought to be uſed in a leſs or greater proportion.

Now conſidering that theſe grey and dun colours may be alſo produced by mixing whites and blacks, and by conſequence differ from perfect whites, not in ſpecies of colours but only in degrees of luminouſneſs, it is manifeſt there is nothing more requiſite to make them perfect whites than to increaſe their light ſufficiently: and on the contrary if by increaſing their light they can be brought to perfect whiteneſs, it will thence alſo follow that they are of the ſame ſpecies of colour with the beſt whites, and differ from them only in quantity of light, and this I tryed as follows. I took the third of the above mentioned grey mixtures (that which was compounded of orpiment, purple, biſe and *viride æris*) and rubbed it thickly upon the floor of my chamber, where the ſun ſhone upon it through the open caſement; and by it, in the ſhadow, I placed a piece of white paper of the ſame bigneſs. Then going from them to the diſtance of 12 or 18 feet, ſo that I could not diſcern the unevenneſs of the ſurface of the powder; nor the little ſhadows let fall from the gritty particles thereof; the powder appeared intenſely white ſo as to tranſcend even the paper it ſelf in whiteneſs, eſpecially if the paper were a little ſhaded from the light of the clouds; and then the paper compared with the powder appeared of ſuch a grey colour as the powder had done before. But by laying the paper where the ſun ſhines through the glaſs window, or by ſhutting the caſement, that the ſun might ſhine through the glaſs upon the powder, and by uſing ſuch other fit means of increaſing or decreaſing the lights wherewith the powder and paper were illuminated, the light wherewith the powder is illuminated may be made ſtronger, in ſuch a due proportion, than the light wherewith the paper is illuminated, that they ſhall both appear exactly alike in whiteneſs. Now if you conſider that this white of the powder in the ſun-ſhine was compounded of the colours which the component powders have in the ſun-ſhine, you muſt acknowledge by this experiment as well as by the former that perfect whiteneſs may be compounded of colours.

The permanent colours of natural bodies explained.

182. The colours of natural bodies ariſe from hence, that ſome of them reflect ſome ſorts of rays, others other ſorts more copiouſly than the reſt. Minium reflects the leaſt refrangible or red-making rays moſt copiouſly and thence appears red. Violets reflect the moſt refrangible moſt copiouſly, and thence have their colour: and ſo of other bodies. Every body reflects the rays of its own colour more copiouſly than the reſt, and from their exceſs and predominance in the reflected light has its colour.

XII. Experiment. Newt. Opt. p. 157.

For if in the homogeneal lights obtained by the 5th experiment, you place bodies of ſeveral colours, you will find as I have done, that every body looks more ſplendid and luminous in the light of its own colour. Cinnaber in the homogeneal red is moſt reſplendent, in the green light it is manifeſtly leſs reſplendent, in the blue light ſtill leſs. Indigo in the violet blue light is moſt reſplendent, and its ſplendor is gradually diminiſhed as it is removed thence by degrees through the green and yellow light to the red. By a leek the green light, and next that the blue and yellow which compound green, are more ſtrongly reflected than the other colours red and violet, and ſo of the reſt. But to make theſe experiments the more manifeſt, ſuch bodies ought to be choſen as have the fulleſt and moſt vivid colours, and two of thoſe bodies are to be compared together. Thus for inſtance, if cinnaber and ultra-marine blue, or ſome other full blue be held together in the red homogeneal light, they will both appear red; but the cinnaber will appear of ſtrongly luminous and reſplendent red, and the ultra-marine blue of a faint obſcure and dark red. And if they be held together in the blue homogeneal light, they will both appear blue; but the ultra-marine will appear of a ſtrongly luminous and reſplent blue, and the cinnaber of a faint and dark blue. Which puts it out of diſpute that the cinnaber reflects the red light much more copiouſly than the ultra-marine doth, and the ultra-marine reflects the blue light much more copiouſly than the cinnaber doth. The ſame experiment may be tried ſucceſſively with red and indigo or with any other two coloured bodies, if due allowance be made for the different ſtrength or weakneſs of their colour and light.

And that this is not only a true reaſon of their colours, but even the only reaſon, may appear farther from this conſideration; that the colour of homogeneal light cannot be changed by the reflection of natural bodies. For if bodies by reflection cannot in the leaſt change the colour of any one ſort of rays, they cannot appear coloured by any other means, than by reflecting thoſe which either are of their own colour, or by mixture muſt produce it.

In tranſparent coloured liquors it is obſervable that their colour uſes to vary with their thickneſs. Thus for inſtance, a red liquor in a conical glaſs held between the light and the eye, looks of a pale and dilute yellow at the bottom where it is thin; and a little higher, where it is thicker grows orange; and where it is ſtill thicker becomes red; and where it is thickeſt the red is deepeſt and darkeſt. For it is to be conceived that ſuch a liquor ſtops the indigo-making, and violet making rays moſt eaſily, the blue-making rays more difficultly, the green-making rays ſtill more difficultly, and the red-making moſt difficultly: and that if the thickneſs of the liquor be only ſuch as ſuffices to ſtop a competent number of the violet-making and indigo-making rays, without diminiſhing much the number of the reſt, the reſt muſt compound a pale yellow; the colour in the ſolar image that lyes in the middle of them, as may be tryed by ſtopping the violet and indigo at the lens in the 9th experiment, and letting the reſt paſs on to be mixed together at the focus. But if the liquor be of ſuch a thickneſs as to ſtop alſo a great number of the blue-making rays and ſome of the green-making, the reſt muſt compound an orange. And where it is ſo thick as to ſtop alſo a great number of the green-making and a conſiderable number of the yellow-making, the reſt muſt begin to compound a red; and this red muſt grow deeper and darker as the yellow-making and orange-making rays are more and more ſtopped by increaſing the thickneſs of the liquor, ſo that few rays beſides the red-making can get through.

If there be two liquors of full colours ſuppoſe a red and a blue, and both of them ſo thick, as ſuffices to make their colours ſufficiently full; though either liquor be ſufficiently tranſparent apart, yet will you not be able to ſee through both together. For if only the red-making rays paſs through one liquor, and only the blue-making through the other, no rays can paſs through both. This Mr. *Hook* tried caſually with glaſs wedges filled with red and blue liquors, and was ſurprized at the unexpected event, the reaſon of it being then unknown [1].

Now whilſt bodies become coloured by reflecting or tranſmitting this or that ſort of rays more copiouſly than the reſt, it is to be conceived that they ſtop and ſtifle in themſelves the rays which they do not reflect or tranſmit. For if gold be foliated and held between your eye and the light, the light looks of a greeniſh blue; and therefore maſſy gold lets into its body the blue-making rays to be reflected to and fro within it till they be ſtopped and ſtifled; whilſt it reflects the yellow-making outwards and thereby looks yellow. And much after the ſame manner that leaf gold is yellow by reflected and blue by tranſmitted light, there are ſome ſorts of liquors, as the tincture of *lignum nephriticum*, and ſome ſorts of glaſs, which tranſmit one ſort of light moſt copiouſly and reflect another ſort, and thereby look of ſeveral colours according to the poſition of the eye to the light. A tranſparent body which looks of any colour by tranſmitted light, may alſo look of the ſame colour by reflected light, the light of that colour being reflected by the farther ſurface of the body[a].

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[1] *Hook's* Micrography p. 73.

[a] Art. 17.

CHAP. VII.

CONCERNING THE CAUSE OF REFRACTION, REFLECTION, INFLECTION AND EMISSION OF LIGHT.

Reflexion not cauſed by the impinging of light upon the medium. Newt. Opt. p. 237.

183. THAT the cauſe of reflection is not the impinging of light on the ſolid or impervious parts of bodies, as is commonly believed, will appear by the following conſiderations. Firſt that in the paſſage of light out of glaſs into air there is a reflection as ſtrong as in its paſſage out of air into glaſs, or rather a little ſtronger, and by many degrees ſtronger than in its paſſage out of glaſs into water. And it ſeems not probable that air ſhould have more reflecting parts than water or glaſs. But if that ſhould poſſibly be ſuppoſed, yet it will avail nothing; for the reflection is as ſtrong or ſtronger when the air is drawn away from the glaſs, by an air-pump, as when it is adjacent to it. Secondly, if light in its paſſage out of glaſs into air be incident more obliquely than an angle of 40 or 41 degrees, it is wholly reflected, if leſs obliquely it is in a great meaſure tranſmitted[b]. Now it is not to be imagined that light at one degree of obliquity ſhould meet with pores enough in the air to tranſmit the greateſt part of it, and at another degree of obliquity ſhould meet with nothing but parts to reflect it wholly: eſpecially conſidering that in its paſſage out of air into glaſs, how oblique ſoever be its incidence, it finds pores enough in the glaſs to tranſmit a great part of it. If any man ſuppoſes that it is not reflected by the air, but by the outmoſt ſuperficial parts of the glaſs, there is ſtill the ſame difficulty: beſides that ſuch a ſuppoſition is unintelligible and will appear to be falſe by applying water behind ſome part of the glaſs inſtead of air. For ſo in a convenient obliquity of the rays ſuppoſe of 45 or 46 degrees, at which they are all reflected where air is adjacent to the glaſs, they ſhall be in a great meaſure tranſmitted where water is adjacent to it. Which argues that their reflection or tranſmiſſion depends on the conſtitution of air and water behind the glaſs and not on the ſtriking of the rays on the parts of the glaſs. Thirdly,

[b] Exp. 4. Art. 173.

Thirdly, if the colours made by a priſm, placed at the entrance of a beam of light into a darkned room, be ſucceſſively caſt on a ſecond priſm placed at a great diſtance from the former, in ſuch manner that they are all alike incident upon it, (as they will be when tranſmitted through the holes in the two boards made uſe of in the 3d experiment,) the ſecond priſm may be ſo inclined to the incident rays, that thoſe which are of a blue colour ſhall be all reflected by it, and yet thoſe of a red pretty copiouſly tranſmitted. Now if reflection be cauſed by the parts of air or glaſs I would aſk why at the ſame obliquity of incidence, the blue ſhould wholly impinge on thoſe parts, ſo as to be all reflected and yet the red find pores enough to be in a great meaſure tranſmitted. Laſtly were the rays of light reflected by impinging on the ſolid parts of bodies, their reflections from poliſhed bodies could not be ſo regular as they are. For in poliſhing glaſs with ſand, putty or tripoli, it cannot be imagined that thoſe ſubſtances can by grating and fretting the glaſs bring all its leaſt particles to an accurate poliſh; ſo that all their ſurfaces ſhall be truly plane or truly ſpherical, and look all the ſame way, ſo as together to compoſe one even ſurface. This manner of poliſhing with powders can do no more than bring the roughneſs of the glaſs to a very fine grain, ſo that the ſcratches and frettings of the ſurface become too ſmall to be viſible. And therefore if light were reflected by impinging upon the ſolid parts of the glaſs, it would be ſcattered as much by the moſt poliſhed glaſs as by the rougheſt. So then it remains a problem how glaſs poliſhed by fretting ſubſtances can reflect light ſo regularly as it does.

Fig. 280.

184. And this problem is ſcarce otherwiſe to be ſolved than by ſaying that the reflection of a ray is effected not by a ſingle point of the reflecting body, but by ſome power of the body which is evenly diffuſed all over its ſurface and by which it acts upon a ray without immediate contact. For that the parts of bodies do act upon light at a diſtance, will appear by the following experiments.

But by an active power diffuſed over its ſurface.

XIII. Experiment. This power acts upon light at a diſtance by attracting and repelling it. Newt. Opt. p. 300.

185. The ſun ſhining into my chamber through a hole a quarter of an inch broad, I placed at the diſtance of two or three feet from the hole a ſheet of paſtboard, which was blacked all over on both ſides, and in the middle of it had a hole about three quarters of an inch ſquare for the light to paſs through. And behind the hole I faſtened to the paſtboard with pitch the blade of a ſharp knife, to intercept ſome part of the light which paſſed through the hole. The planes of the paſtboard and of the knife were parallel to one another and perpendicular to the rays. And when they were ſo placed that none of the ſun's light fell upon the paſtboard, but all of it paſſed through the hole to the knife, and there part of it fell upon the blade of the knife, and part of it paſſed by its edge; I let this part of the light, which paſſed by, fall on a white paper two or three feet beyond the knife, and there ſaw two ſtreams of faint light ſhoot out

both ways from the beam of light into the ſhadow, like the tails of comets. But becauſe the ſun's direct light by its brightneſs upon the paper obſcured theſe faint ſtreams ſo that I could ſcarce ſee them, I made a little hole in the midſt of the paper for that light to paſs through, and fall upon a black cloth behind it, and then I ſaw the two ſtreams plainly. They were like one another and pretty nearly equal in length and breadth, and quantity of light. Their light at that end next the ſun's direct light was pretty ſtrong for the ſpace of about a quarter of an inch or half an inch, and in all its progreſs from that direct light decreaſed gradually till it became inſenſible. The whole length of either of theſe ſtreams meaſured upon the paper, at the diſtance of three feet from the knife, was about ſix or eight inches; ſo that it ſubtended an angle at the edge of the knife of 10 or 12, or at moſt 14 degrees.

XIV. Experiment. I placed another knife by this, ſo that their edges might be parallel and look towards one another, and that the beam of light might fall upon both knives and ſome part of it paſs between their edges. And when the diſtance of their edges was about the 400th part of an inch, the ſtream parted in the middle and left a ſhadow between the two parts. This ſhadow was ſo black and dark that all the light which paſſed between the two knives ſeemed to be bent and to be turned aſide to the one hand and to the other. And as the knives ſtill approached one another the ſhadow grew broader, and the ſtreams ſhorter at their inward ends next the ſhadow, until upon the contact of the knives the whole light vaniſhed and left its place to the ſhadow. And hence I gather that the light which is leaſt bent, and goes to the inward ends of the ſtreams, paſſes by the edges of the knives at the greateſt diſtance, and this diſtance when the ſhadow begins to appear between the ſtreams is about the 800th part of an inch. And the light which paſſes by the edges of the knives at diſtances ſtill leſs and leſs is more and more bent, and goes to thoſe parts of the ſtreams which are farther and farther from the direct light. Becauſe when the knives approach one another till they touch, thoſe parts of the ſtreams vaniſh laſt which are fartheſt from the direct light.

Our author has made it appear from theſe and ſome other experiments, that bodies act upon light in ſome circumſtances by an attractive and in others by a repulſive power. For he found that the ſhadows of hairs, threds, pins, ſtraws, and ſuch like ſlender ſubſtances, placed in a ſlender beam of light let into a dark room, were conſiderably broader than they ought to be, if the rays of light paſſed on by theſe bodies in right lines. Particularly he found that the ſhadow of a hair of a man's head, at the diſtance of 10 feet from the hair, was 35 times broader than the hair it ſelf[a]. And of theſe attractive and repulſive powers he declares his ſentiments more fully in theſe words[b]. Since metals diſſolved in acids attract but a ſmall quantity of the acid, their attractive force can reach but to a ſmall diſtance

[a] Newt. Opt. p. 293.

[b] Ibid. p. 370.

diſtance from them. And as in algebra where affirmative quantities vaniſh and ceaſe there negative ones begin, ſo in mechanicks where attraction ceaſes there a repulſive virtue ought to ſucceed. And that there is ſuch a virtue ſeems to follow from the reflections and inflections of the rays of light. For the rays are repelled by bodies in both theſe caſes without the immediate contact of the reflecting or inflecting body[a]. It appears alſo from the emiſſion of light: the ray ſo ſoon as it is ſhaken off from a ſhining body, by the vibrating motion of the parts of the body, and gets beyond the reach of attraction, being driven away with exceeding great velocity. For that force which is ſufficient to turn it back in reflection, may be ſufficient to emit it. It ſeems alſo to follow from the production of air and vapours. The particles when they are ſhaken off from bodies by heat or fermentation, ſo ſoon as they are beyond the reach of the attraction of the body, receding from it and alſo from one another with great ſtrength, and keeping at a diſtance ſo as ſometimes to take up a million of times more ſpace than they did before in the form of a denſe body. Which vaſt contraction and expanſion ſeems unintelligible by feigning the particles of air to be ſpringy and ramous, or rolled up like hoops or by any other means than by a repulſive power. From this power it ſeems to be that flies walk upon water without wetting their feet, and that the object-glaſſes of long teleſcopes lye upon one another without touching; and that two poliſhed marbles which by immediate contact ſtick together, are difficultly brought ſo cloſe together as to ſtick.

[a] Experiment 13, 14.

And is infinitely ſtronger than the power of gravity.

186. That this power which acts upon light is infinitely ſtronger than the power of gravity will appear by the following argument. Sir *Iſaac Newton* has demonſtrated that all bodies attract one another by the force of gravity, and that the attractive forces of two homogeneal ſpheres, upon particles of matter placed very near their ſurfaces, are to each other in proportion as the diameters of the ſpheres[1]. That is to ſay, if a refracting medium be ſpherical and of the ſame denſity as the earth, the earths force of attraction near its ſurface, will exceed the mediums force near its ſurface, as much as the diameter of the earth exceeds the diameter of the medium; or almoſt infinitely with reſpect to human conceptions. Yet we obſerve that a cannon-ball, juſt ſhot from the mouth of the cannon, is ſcarce ſenſibly deflected towards the earth by its attraction; and the leaſt particle of the ball, if it was ſeparate from the reſt, would be no more deflected than the whole; becauſe gravity makes bodies of all ſorts and ſizes deſcend with the ſame ſwiftneſs, by affecting them alike whether joined or ſeparated. Therefore a particle of light which moves, I may ſay, infinitely quicker than a cannon-ball[2], would be infinitely

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[1] Princip. lib. 1. prop. 74. cor. 2. & lib. 3. prop. 8.

[2] See velocity of light in the Index of this book.

leſs bent than the particle of the ball by the attraction of the whole earth, and ſtill infinitely leſs, than this laſt bending, by the attraction of the ſpherical medium, which was ſhewn to be infinitely weaker than that of the earth. But in fact we find it is very ſenſibly bent or refracted; and therefore it muſt be affected by ſome other power of the medium, which near its ſurface is infinitely ſtronger than the power of gravity.

And decreaſes much quicker.

187. It is difficult to determine the exact law of this refractive power, or the degrees of its force at given diſtances from the refracting ſurface. However ſince we find that the effects of gravity, which decreaſe as the ſquares of the diſtances from the center increaſe, are very ſenſible at great diſtances, we may conclude that the refractive power of a medium, which at its ſurface we find is infinitely ſtronger than gravity, and yet vaniſhes at a very ſmall diſtance from it[a], decreaſes much quicker or in a greater proportion than gravity does.

This one power both refracts and reflects light. Newt. Opt. p. 244.

188. It is reaſonable to conclude that bodies reflect and refract light by one and the ſame power variouſly exerciſed in various circumſtances; becauſe when light goes out of glaſs into air as obliquely as it can poſſibly do, if its incidence be made ſtill more oblique, it becomes totally reflected[b]: (for the power of the glaſs after it has reflected the light as obliquely as is poſſible, if the incidence be ſtill made more oblique becomes too ſtrong to let any of its rays go through and by conſequence cauſes total reflections:) And for this other reaſon, that thoſe ſurfaces of tranſparent bodies which have the greateſt refracting power, reflect the greateſt quantity of light, as will be ſhewn in the next chapter.

Its forces in different bodies are as their denſities nearly. Newt. Opt. p. 245.

189. From the different ratios of the ſines of incidence and refraction in a great many different bodies, our author has alſo collected that the forces of bodies to reflect and refract light are very nearly proportionable to their denſities, excepting that unctuous and ſulphureous bodies refract more than others of the ſame denſity. Whence, he ſays, it ſeems rational to attribute the refractive power of all bodies chiefly, if not wholly to the ſulphurous, oily particles with which they abound. For it is probable that all bodies abound more or leſs with ſulphurs. And as light congregated by a burning glaſs acts moſt upon ſulphureous bodies to turn them into fire and flame, ſo ſince all action is mutual, ſulphurs ought to act moſt upon light. For that the action between light and bodies is mutual, may appear from this conſideration; that the denſeſt bodies which refract and reflect light moſt ſtrongly grow hotteſt in the ſummer ſun, by the action of the refracted or reflected light.

Refractive denſities what.

If bodies be conceived to have certain denſities exactly proportionable to their refractive powers, theſe may be called their refractive denſities.

This force acts in lines perpendicular to the refracting ſurface.

190. The direction of the refractive force of a medium, acting upon particles of light, is every where perpendicular to the refracting ſurface. For whether this force be a real attraction, or whether it be an impulſe

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[a] Art. 185.

[b] Art. 173.

upon light, cauſed by the ſpring or elaſtick power of a ſubtil fluid which pervades the medium, and being gradually denſer without than within it, may impel the light towards the medium by its greater elaſticity without than within[a]; be this as you pleaſe, yet if the medium be uniform in all its parts, its immediate power upon the light it ſelf, or upon the ſubtil fluid which acts upon it, will be equally ſtrong in every point of a plane drawn parallel to the refracting ſurface; though its ſtrength may be different in the next parallel plane, and ſtill different in the next, and ſo on as far as that power is extended on each ſide of the ſurface of the medium. The extent of this power will therefore be terminated by two planes, parallel to one another and to the refracting ſurface; and the ſpace between them may be called the ſpace of activity, whether the power attracts or repels. This being premiſed, I ſay the force of the medium will act upon light, either in attracting or repelling it, in lines perpendicular to its ſurface. For let *p* be a particle of light acted upon by any uniform power in the line *de* parallel to the refracting ſurface *AB*, *pc* a line perpendicular to thoſe parallels, cutting *de* in *c*; it is evident that the force of the power at *c* will move the particle *p* in the line *pc*; and taking any two points *d*, *e* at equal diſtances on each ſide of *c*, the powers at *d* and *e* being equal and acting at equal diſtances, *pd*, *pe*, equally inclined to *pc*, cannot move *p* in any direction but that of *pc*; and what has been ſaid of the equal powers in the line *de* is applicable to the powers in every line drawn parallel to *AB*, that is to the whole power of the refracting medium.

Space of activity what.

Fig. 286, 287.

191. Now when a ray of light falls perpendicularly upon the ſpace of activity, its particles will be accelerated or retarded in the ſame perpendicular direction, according as the power of the medium acts with or againſt the courſe of their motion; and when the particles are got through that ſpace they will proceed with an uniform velocity. But if a ray *op* or *sr* falls obliquely upon the ſpace of activity *klmn*, the force of the medium now acting ſideways or obliquely upon the particles, will bend their courſe into a curve *pqr*, during their paſſage through that ſpace. For as light has this property in common with all other bodies, of moving ſtraight forwards, while its motion is not diſturbed by any oblique force, ſo when it is diſturbed, we may reaſonably conclude, it will follow thoſe other laws of motion, to which all other bodies are equally ſubject. The force of the medium acting ſideways upon its oblique courſe, will therefore draw it perpetually out of one direction into another. But having paſſed through the ſpace of activity, it will then proceed ſtraight forwards; for being attracted or impelled every way alike, or elſe not at all if it be in empty ſpace, it will have the ſame freedom of motion in both caſes: juſt as an animal ſurrounded with air, though violently preſſed on every ſide, feels no conſtraint, but has an equal facility of moving in any di-

The manner of its operation in cauſing refractions and reflections.

Fig. 288.

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[a] Newt. Opt. p. 323. &c.

rection. Thus we ſee that the refraction of light is performed in the ſame manner as if a ſtone was thrown in the direction *op*, and its courſe was bent into a curve *pqr* by its gravity; or being thrown the contrary way in the direction *sr*, it was bent into the curve *rqp* in aſcending: and ſuppoſing the attraction of the earth to reach no higher than the line *kl* the ſtone would from thence proceed in a ſtraight line *po*. Now the gravity of the ſtone may be ſo great, or the force of projection ſo weak, or the direction of the motion ſo oblique to the horizontal line *kl*, that it cannot aſcend ſo high as this line. In this caſe the ſtone will deſcend from the higheſt point of its courſe by the ſame degrees of curvity with which it aſcended; and if its gravity be ſuppoſed to ceaſe in all places below the line *mn*, the ſtone will go on in the direction of the laſt particle of the curve produced. This is a parallel caſe to that of reflections from the farther ſurface of denſe mediums, when the incident ray is ſo much inclined to that ſurface as to be pulled back into the ſame medium. Hitherto I have ſuppoſed the refracting medium to be contiguous to empty ſpace; but the manner of reflection and refraction is the ſame at the common ſurface of any two mediums. For ſince the ſeparate forces of the mediums act in the ſame lines, perpendicular to their common ſurface[a], and in contrary directions; the light will be affected with the difference of thoſe forces in the ſame manner as before. And if the mediums have equal forces they will ballance each other, without cauſing any reflection or refraction at all. It has been obſerved already that the perpendicular breadth of the ſpace of activity is exceeding ſmall, and conſequently in phyſical experiments the incurvation of the ray may ſtill be conſidered as performed in a phyſical point.

Fig. 289.

[a] Art. 190.

And in cauſing the different refrangibility of rays. Newt. Opt. p. 347. Fig. 290.

192. According to this theory nothing more is requiſite for producing all the variety of colours and degrees of refrangibility, than that the rays of light be bodies of different ſizes; the leaſt of which may make violet, the weakeſt and darkeſt of the colours, and be more eaſily diverted by refracting ſurfaces from its right courſe; and the reſt, as they are bigger and bigger, may make the ſtronger and more lucid colours, blue, green, yellow and red; and be more and more difficultly diverted. For particles of different ſizes, that fall upon the ſpace of activity *klmn* in the line *op*, having different forces, may deſcribe different curves, as *pa*, *pb*, *pc*; and conſequently will emerge from that ſpace in different angles.

And in cauſing the angles of reflection to be the ſame in all ſorts of rays. Fig. 290.

193. Thus may heterogeneal particles diverge from one another by refraction, though not by reflection. For if the line of their incidence *op* be ſo oblique to the ſpace of attraction *klmn*, that all the particles are pulled back into the ſame medium, they will return in parallel lines *rs*, *tv*, *xy*, &c. inclined to that ſpace in the ſame angles as the line of incidence *op* is inclined to it. Juſt as ſeveral balls of different ſizes, ſhot with different forces out of a cannon *op* in any fixt poſition, will deſcribe dif-

ferent curves, as *pdr*, *pet*, *pfx*, &c. yet in returning to the ground they will all ſtrike upon it in equal angles at *r*, *t*, *x*, &c. every one being equal to the angle of elevation at *p*. Now ſince the ſpace of attraction is exceeding thin, the parallel lines *rs*, *tv*, &c. will be ſo cloſe together that the ſenſe cannot perceive a diſtinct ſenſation of the ſeparated particles, and conſequently the reflected and incident light will appear of the ſame colour. And when the incident light conſiſts of ſeveral rays, though the particles in each ray may be a little ſeparated after reflection, and proceed in different lines, yet thoſe ſeveral lines will be mixt together, and conſequently the reflected light will appear white or of the ſame colour as the incident light.

And in cauſing reflections from opake bodies and from the firſt ſurface of tranſparent ones. Fig. 291.

194. By what I have quoted from Sir *Iſaac Newton* in the 185th article, his notion of the cauſe and manner of reflection from opake bodies, and from the firſt ſurface of tranſparent bodies, ſeems to be this that follows. Let the attractive power of the denſe medium *ABCD* end at the line *kl*, and there let the repulſive power begin, and let it end at the parallel line *bi*; and when a ray *op* falls from air upon the ſpace of repulſion *bikl*, it will be perpetually diverted from one direction into another by the oppoſition of the repulſive force, and ſo will deſcribe a curve *pqr*, till it emerges from that ſpace in the ſame angle at *r* with which it immerged at *p*, and then it will proceed in a right line *rs*. This will be the courſe of the ray if its progreſſive force be but weak, or the repulſive force be ſo ſtrong as to hinder it from entering the ſpace of attraction *klmn*: For if it enters this ſpace, inſtead of being reflected, it will be refracted into the denſe medium. And in reality ſome part of the incident light is always reflected and ſome refracted at all tranſparent ſurfaces; the cauſe of which our author has alſo conſidered[a].

Hence it ſeems to follow that the repulſive power of a denſe medium is leſs extended or elſe weaker than the attractive. For if the bending of a ray by the repulſive power, was not leſs than the contrary bending made by the attractive, the refraction into a denſe medium could not always be made towards the perpendicular, as it always is. We may alſo obſerve that a refracted ray, in its paſſage through the ſurface of a tranſparent medium, is bent backward and forward with a motion like that of an eel; and our author takes notice of the ſame ſort of motion in its paſſage by the edges and ſides of bodies. It follows alſo that the repulſive power does not extend to a ſenſible diſtance from the medium; for if it did, it would be diſcovered by a ſenſible incurvation of the ray throughout that extent; contrary to experience.

The diſtinguiſhing character of Sir *Iſaac Newton*'s philoſophy.

195. Thus ſays our author will nature be very conformable to her ſelf, in performing all the great motions of the heavenly bodies by the attraction of gravity, which intercedes thoſe bodies, and almoſt all the ſmall ones of their particles by ſome other attractive and repelling powers,

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[a] Newt. Opt. p. 253.

which intercede the particles[a]. Since bodies at reſt, or put in motion, cannot alter their ſtate of reſt, or the direction of their motion; it follows that bodies which move in curves, as the planets do, are continually drawn or impelled out of one direction into another by ſome power acting conſtantly upon them: juſt as bodies projected in the air are drawn into curves by the power of gravity. Without aſſuming any other principles, our author has ſhewn by the ſtricteſt reaſoning, what effects this power muſt produce among comets, planets and ſatellits once put into motion; as what would be the ſhapes and poſitions of their orbits; with what velocities they would move in different places; what proportions there would be between the times of their periods about a central body, with reſpect to their diſtances from it; what diſturbances they would cauſe in each others motions, and in the flux and reflux of the ſea, and the like. In a word he has exactly deſcribed all the particulars of a planetary ſyſtem. Then by comparing this ſyſtem with facts and obſervations, made upon the real motions, periods and diſtances of the planets and comets, he has ſhewn ſo exact an agreement between them, not in groſs but quantity for quantity, in a thouſand particulars, that whatever differences appeared at firſt, are always found to be leſs and leſs by more accurate obſervations. Now as ſo exact a conformity of reaſon and experience is the greateſt evidence we can poſſibly have, that this explication of the mechaniſm of the world is a true explication; and as it is the diſtinguiſhing character of his ſyſtem, which all the philoſophers that lived before him could never give of any of theirs, for want of ſufficient ſkill in geometry; ſo likewiſe he has given us the ſame evidence for the truth of his theory of light. For inſtance, having ſhewn by experiments that light and bodies affect one another at a diſtance, by ſome power that intercedes them; whatever be the law of this power, he has proved mathematically, by two different methods, that in all refractions the ſine of incidence muſt be to the ſine of refraction in a given ratio, and in reflections that the angles and ſines are equal[b]. I have made this compariſon more at large to rectify an opinion ſtill current, even amongſt men of learning; who obſerving that various ſyſtems of philoſophy in ſucceſſive ages have riſen and ſunk in the opinions of men, like modes and faſhions, and not conſidering in what reſpect our author's ſyſtem is plainly diſtinguiſhed from that of all others, are apt to think that his alſo in time will ſuffer the ſame fate. But it muſt be allowed, ſo long as there is a conſtancy in the nature of things, and a mutual agreement of reaſon and experience, that a ſyſtem founded entirely upon this alone and nothing elſe, can never be changed, nor be leſs eternal than the world and truth it ſelf.

I will conclude this book with our author's diſcoveries concerning the particular conſtitutions of bodies; which render ſome of them tranſparent, others opake and coloured.

[a] Newt. Opt. p. 372.

[b] Opt. p. 68. Princip. lib. 1. prop. 94.

CHAP. VIII.

CONCERNING TRANSPARENCY, OPACITY AND COLOURS IN BODIES.

Stronger and weaker reflections how cauſed. Newt. Opt. p. 220.

196. THOSE ſuperficies of tranſparent bodies reflect the greateſt quantity of light which have the greateſt refractive power; that is which intercede mediums that differ moſt in their refractive denſities[a]: and in the confines of equally refracting mediums there is no reflection. The analogy between refraction and reflection will appear by conſidering, that when light paſſes obliquely out of one medium into another which refracts from the perpendicular, the greater is the difference of their refractive denſities, the leſs obliquity of incidence is requiſite to cauſe a total reflection[b]. Thoſe ſuperficies therefore which refract moſt, do ſooneſt reflect all the light which is incident upon them, and ſo muſt be allowed moſt ſtrongly reflective. But the truth of this propoſition will farther appear by obſerving, that in the ſuperficies interceding two tranſparent mediums (ſuch as are air, water, oil, common glaſs, cryſtal, metalline glaſs, iſland glaſſes, white tranſparent arſenick, diamonds, &c.) the reflection is ſtronger or weaker accordingly as the ſuperficies hath a greater or leſs refractive power. For in the confine of air and ſal-gem it is ſtronger than in the confine of air and water, and ſtill ſtronger in the confine of air and common glaſs or cryſtal, and ſtronger in the confine of air and a diamond. If any of theſe and ſuch like tranſparent ſolids, be immerged in water, its reflection becomes much weaker than before, and ſtill weaker if they be immerged in the more ſtrongly refracting liquors of well rectified oil of vitriol or ſpirit of turpentine. If water be diſtinguiſhed into two parts by an imaginary ſurface, the reflection in the confine of theſe two parts is none at all; in the confine of water and ice it is very little; and in that of water and oil it is ſomething greater; in that of water and ſal-gem ſtill greater, and in that of water and glaſs or cryſtal or other denſer ſubſtances ſtill greater, accordingly as thoſe mediums differ more or leſs in their refracting powers. Hence in the confine of common glaſs and cryſtal there ought to be a weak reflection, and a ſtronger reflection in the confine of common and metalline glaſs, though I have not yet tried this. But in the confine of two glaſſes of equal denſities, as of two object-glaſſes of long teleſcopes preſſed gently together, there is not any ſenſible reflection. For objects may be ſeen by rays obliquely tranſmitted through a round black ſpot where the glaſſes touch one another, but not through other places where the light is reflected at the interval between the glaſſes. And the ſame may be underſtood of the ſuperficies interceding two cryſtals, or two liquors, in which no reflection is cauſed. So then the reaſon why uniform pellu-

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[a] Art. 189.

[b] Art. 171.

cid mediums, ſuch as water, glaſs, or cryſtal, have no ſenſible reflection, but in their external ſuperficies, where they are adjacent to other mediums of a different denſity, is becauſe all their contiguous parts have one and the ſame degree of denſity: or this uniform denſity of their contiguous parts is a neceſſary condition of the tranſparency of the whole.

Opacity cauſed by a multitude of internal reflections. Newt. Opt. p. 222.

197. The leaſt parts of almoſt all natural bodies are in ſome meaſure tranſparent: and the opacity of thoſe bodies ariſeth from the multitude of reflections cauſed in their internal parts. That this is ſo hath been obſerved by others, and will eaſily be granted by them that have been converſant with microſcopes. And it may alſo be tried by applying any ſubſtance to a hole, through which ſome light is immitted into a dark room. For how opake ſoever that ſubſtance may ſeem in the open air, it will by that means appear very manifeſtly tranſparent, if it be of a ſufficient thinneſs. Only white metalline bodies muſt be excepted, which by reaſon of their exceſſive denſity ſeem to reflect almoſt all the light incident on their firſt ſuperficies; unleſs by ſolution in menſtruums they be reduced into very ſmall particles, and then they become tranſparent.

The conſtitution of opake and coloured bodies what. Newt. Opt. p. 223.

198. Between the parts of opake and coloured bodies are many ſpaces, either empty or repleniſhed with mediums of other denſities; as water between the tinging corpuſcles wherewith a liquor is impregnated; air between the aqueous globules that conſtitute clouds or miſts; and for the moſt part ſpaces void of both air and water, but yet perhaps not wholly void of all ſubſtance, between the parts of hard bodies. The truth of this is evident by the two preceding articles. For by the latter article there are many reflections made by the internal parts of bodies, which by the former article would not happen if the parts of theſe bodies were continued, without any ſuch interſtices between them: becauſe reflections are cauſed only in ſuperficies which intercede mediums of a different denſity by article 196.

But farther, that this diſcontinuity of parts is the principal cauſe of the opacity of bodies, will appear by conſidering, that opake ſubſtances become tranſparent by filling their pores with any ſubſtance of equal or almoſt equal denſities with their parts. Thus paper dipped in water or oil, the *oculus mundi* ſtone ſteeped in water, linen cloth oiled or varniſhed, and many other ſubſtances ſoaked in ſuch liquors as will intimately pervade their little pores, or ſeparating parts, become by that means more tranſparent than otherwiſe. So on the contrary, the moſt tranſparent ſubſtances may by evacuating their pores, or ſeparating parts, be rendered ſufficiently opake; as ſalts or wet paper or the *oculus mundi* ſtone by being dried; horn by being ſcraped; glaſs by being reduced to powder or otherwiſe flawed; turpentine by being ſtirred about with water till they mix imperfectly; and water by being formed into many ſmall bubbles, either alone in the form of froth, or by ſhaking it together with oil of turpentine or oil of olive or with ſome other convenient liquor, with which it will not perfectly incorporate.

The conſtitution of tranſparent bodies what. Newt. Opt. p. 225.

199. The parts of bodies and their interſtices muſt not be leſs than of ſome definite bigneſs, to render them opake and coloured. For the opakeſt bodies, if their parts be ſubtily divided, (as metals by being diſſolved in acid menſtruums &c.) become perfectly tranſparent; and at the ſuperficies of the object-glaſſes, mentioned in the 196th article, where they were very near to one another though they did not abſolutely touch, there was no ſenſible reflection. And likewiſe if a bubble be blown with water firſt made tenacious by diſſolving a little ſoap in it, and be covered with a clear glaſs, to defend it from being agitated by the external air, and be ſuffered to reſt a while, till by the continual ſubſiding of the water it becomes very thin; at the top where it is thinneſt, there will grow a round, black, ſpot (like that between the object-glaſſes) which will continually dilate it ſelf more and more till the bubble breaks; now this ſpot appears black and tranſparent for want of a ſenſible reflection, whereas the ſides of the bubble which are thicker than the top appear coloured and opake by a ſtrong reflection.

On theſe grounds I perceive it is that water, ſalt, glaſs, ſtones and ſuch like ſubſtances are tranſparent. For upon diverſe conſiderations they ſeem to be as full of pores or interſtices between their parts as other bodies are, but yet their parts and interſtices to be too ſmall to cauſe reflections in their common ſurfaces.

Rings of colours in water-bubbles &c. explained. Newt. Opt. p. 168.

200. The black ſpots at the top of the water-bubble, and in the middle of the object-glaſſes compreſſed together, are always ſurrounded by a multitude of concentrick rings of all ſorts of colours: and as the colour in every ring is the ſame quite round its circumference, and different in different rings, ſo it is manifeſt (from the ſpherical figure both of the object-glaſſes and of the bubble of water, and from the uniform gravity of the particles of the water ſubſiding gradually on all ſides from the top to the bottom) that the thickneſſes, both of the plate of air between the glaſſes and of the water-bubble, are alſo the ſame in every part of the ſame ring, and different in different rings. Which ſhews that the particular colour of any ring depends upon a particular thickneſs of the plate of air or ſhell of water, where the incident light of the open air is reflected to the eye. Rings of colours do alſo appear by light tranſmitted through the water-bubble, and through the object-glaſſes, held between the eye and the light, but their colours are different from thoſe that appear in the ſame places, by reflected light.

Theſe are the general appearances of the rings in the open air; but when homogeneal light is caſt by a priſm upon the object-glaſſes in a dark room, the colour of all the rings, ſeen by light reflected from the glaſſes, is the ſame as of the light thrown upon them; and in the intervals between the coloured rings other dark rings appear like the ſpot in the middle; through all which the incident light is tranſmitted, and forms other intermediate rings of the ſame colour upon a white paper held behind the glaſſes, as repreſented in the figure. Fig. 294. It is alſo to be obſerved that the diameters, breadths and intervals of rings of homogeneal lights, of different colours, are all different; thoſe made by homogeneal red being the largeſt, and by homogeneal violet the leaſt; and thoſe of the intermediate priſmatick colours are of intermediate ſizes. And from hence the origin of the different coloured rings in the open air is manifeſt; namely, that the air between the glaſſes, according to its various thickneſs, is diſpoſed in ſome places to reflect and in others to tranſmit the light of any one colour; and in the ſame place to reflect that of one colour where it tranſmits that of another. The appearances are the ſame when water is between the object-glaſſes, only the rings are ſmaller.

The conſtitution of coloured bodies farther explained. Newt. Opt. p. 226.

201. The tranſparent parts of bodies according to their ſeveral ſizes reflect rays of one colour and tranſmit thoſe of another; on the ſame grounds that thin plates or bubbles do reflect or tranſmit thoſe rays. And this I take to be the ground of all their colours. For if a thinned or plated body, which being of an even thickneſs appears all over of one uniform colour, ſhould be ſlit into threads or broken into fragments, I ſee no reaſon why every thread or fragment ſhould not keep its colour; and by conſequence why a heap of thoſe threads or fragments ſhould not conſtitute a maſs or powder of the ſame colour which the plate exhibited before it was broken. And the parts of all natural bodies being like ſo many fragments of a plate muſt on the ſame grounds exhibit the ſame colour.

Now that they do ſo will appear by the affinity of their properties. The fineſt coloured feathers of ſome birds, and particularly thoſe of peacocks tails, do in the very ſame part of the feather appear of ſeveral colours in ſeveral poſitions of the eye; after the ſame manner as the rings do in the water-bubble and between the object-glaſſes; and therefore their colours ariſe from the thinneſs of the tranſparent parts of the feathers; that is from the ſlenderneſs of the very fine hairs, or *capillamenta*, which grow out of the ſides of the groſſer lateral branches or fibres of thoſe feathers. And to the ſame purpoſe it is that the webs of ſome ſpiders, by being ſpun very fine, have appeared coloured, as ſome have obſerved; and that the coloured fibres of ſome ſilks by varying the poſition of the eye do vary their colours. Alſo the colours of ſilks, cloths and other ſubſtances, which water or oil can intimately penetrate, become more faint and obſcure by being immerged in thoſe liquors; and recover their vigour again by being dried; much after the ſame manner as the colours of thin plates of muſcovy glaſs become more faint and languid by wetting them, and alſo thoſe of the rings by letting water creep in between the object-glaſſes. Leaf gold and ſome ſorts of painted glaſs, and the infuſion of

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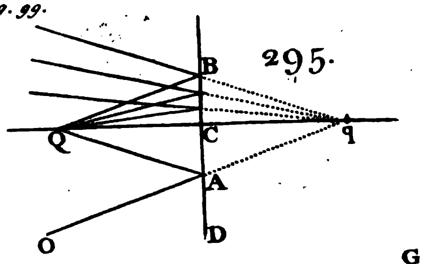

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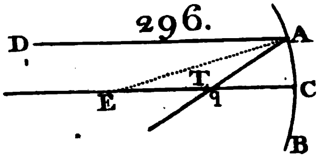

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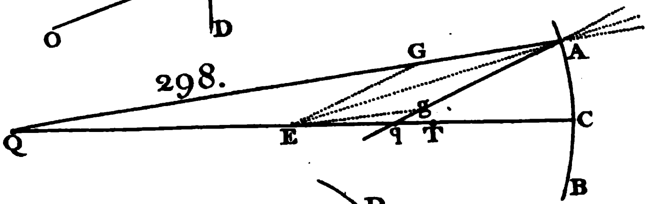

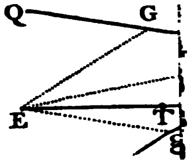

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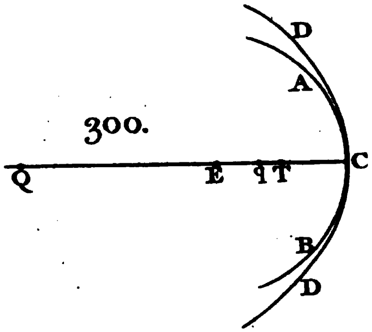

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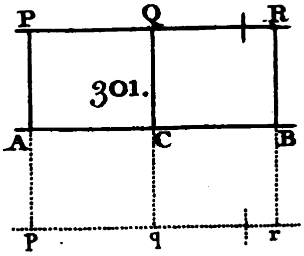

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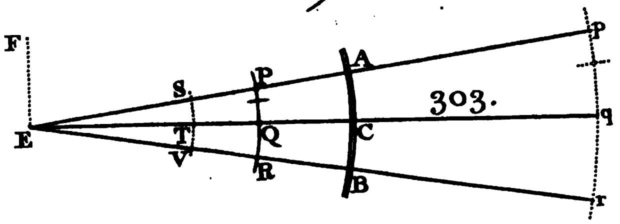

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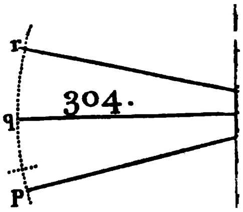

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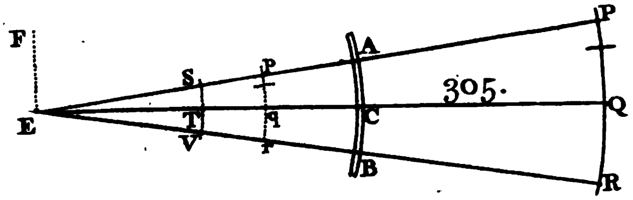

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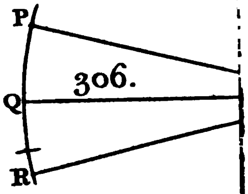

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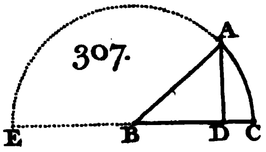

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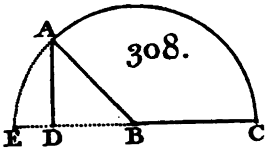

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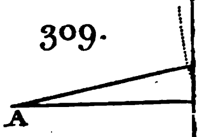

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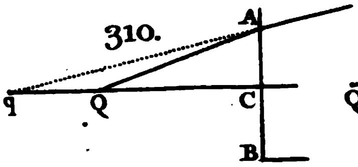

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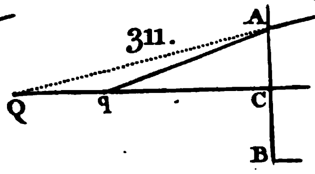

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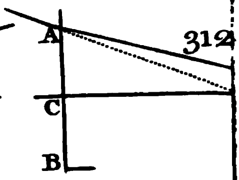

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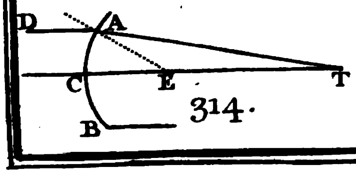

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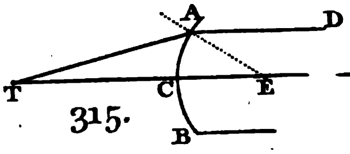

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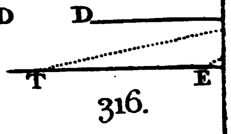

lignum nephriticum, reflect one colour and tranſmit another, as the rings in the water-bubbles and between the object-glaſſes do in the very ſame places. and ſome of thoſe coloured powders which painters uſe, may have their colours a little changed by being very elaborately and finely ground. Where I ſee not what can be juſtly pretended for thoſe changes, beſides the breaking of their parts into leſs parts by that contrition, after the ſame manner that the colour of a thin plate is changed by varying its thickneſs. For which reaſon it is that the coloured flowers of plants and vegetables by being bruiſed uſually become more tranſparent than before, or at leaſt in ſome degree or other change their colours. Nor is it much leſs to my purpoſe, that by mixing divers liquors very odd and remarkable productions and changes of colours may be effected [1]; of which no cauſe can be more obvious and rational than that the ſaline corpuſcles of one liquor, do variouſly act upon or unite with the tinging corpuſcles of another; ſo as to make them ſwell, or ſhrink, (whereby not only their bulk but their denſity may alſo be changed;) or to divide them into ſmaller corpuſcles, (whereby a coloured liquor may become tranſparent [a];) a Art. 199.
or to make many of them aſſociate into one cluſter, whereby two tranſparent liquors may compoſe a coloured one [b]. For we ſee how apt thoſe b Art. 198.
ſaline menſtruums are to penetrate and diſſolve ſubſtances to which they are applied, and ſome of them to precipitate what others diſſolve. In like manner, if we conſider the various phænomena of the atmoſphere, we may obſerve, that when vapours are firſt raiſed, they hinder not the tranſparency of the air, being divided into parts too ſmall to cauſe any reflection in their ſuperficies [c]. But when in order to compoſe drops of rain c Art. 199.
they begin to coaleſce and conſtitute globules of all intermediate ſizes, theſe globules when they become of a convenient ſize to reflect ſome ſorts of rays and tranſmit others, may conſtitute clouds of various colours according to their ſizes. And I ſee not what can be rationally conceived, in ſo tranſparent a ſubſtance as water, for the production of theſe colours beſides the various ſizes of its fluid and globular parcels.

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[1] See *Boyle* upon colours, experim. 20. p. 245.

A
COMPLEAT SYSTEM
OF
OPTICKS.

BOOK II.

A
Mathematical Treatiſe.

CHAPTER I.

To find the focus of rays reflected from any given ſurface.

PROPOSITION I.

Fig. 295. 202. LET ACB *be a reflecting plane, and* Q *the focus of the incident rays, and* QC *a perpendicular to that plane; and if this perpendicular be produced to* q, *ſo that* qC *be equal to* QC, *the point* q *ſhall be the focus of the reflected rays.*

For let *QA* be any incident ray; draw *qA* and produce it towards *O*, and *CA* towards *D*. Then becauſe *Cq* is made equal to *CQ*, the triangles
a 8th Ax. Eucl. *CAq*, *CAQ* will be equal[a]. And conſequently the angle *DAO*, which is equal to the oppoſite angle *CAq*, is alſo equal to the angle *CAQ*. There-
b Art. 9. fore *AO* is the reflected ray[b]. *Q. E. D.*

c Art. 11. 203. *Corol.* Hence all the rays that flow towards *q*, will flow to *Q* after reflection[c].

LEMMA.

204. *Quantities and their proportions, which ſo approach to a ſtate of equality as to become equal at laſt, may be taken for equal in a ſtate immediately preceding the laſt; and alſo in a ſtate ſomewhat remote from the laſt without ſenſible error in phyſical ſubjects: and the ſame may be ſaid of figures which continually approach to a ſtate of ſimilitude; eſpecially if theſe errors, when computed, are found inconſiderable.*

The meaning of the lemma will appear very plain when applied to the following propoſitions.

PROPOSITION II.

Fig. 296, 297. 205. *When parallel rays as* DA, EC *fall almoſt perpendicularly upon a ſpherical ſurface* ACB, *the focus,* T, *of the reflected rays will biſect that ſemidiameter* EC, *which is parallel to the incident rays.*

For drawing *EA*, it will be perpendicular to the ſpherical ſurface at *A*, and ſince *EC* is in the ſame plane as the angle of incidence *DAE*, the reflected ray *Aq* (produced backwards in fig. 297.) will meet *EC* ſomewhere in *q*[a]; ſo that the angle of reflection *EAq* may be equal[b] to the angle of incidence *EAD* or to the alternate angle *AEq*. The two ſides *Aq*, *Eq* of the triangle *AqE* are therefore equal to each other[c]; and conſequently each of them greater than half the third ſide *EA*, or than *ET* by conſtruction. But as the point of incidence at *A* approaches towards *C* the lines *Eq*, *ET* continually approach towards equality, and become equal when the triangle *AEq* is vaniſhing: and ſo the focus of rays falling almoſt perpendicularly on the ſurface, or the neareſt to the point *C*, is to be reckoned at *T*[d]. *Q. E. D.*

206. *Corol.* Hence if *T* be the focus of the incident rays, the reflected ones will go parallel to the line *TE*[e].

PROPOSITION III.

Fig. 298, 299.

207. *Let* ACB *be a reflecting ſurface of any ſphere whoſe center is* E. *Biſect any radius thereof, ſuppoſe* EC, *in* T; *and if in this radius, on the ſame ſide of the point* T, *you take the points* Q *and* q, *ſo that* TQ, TE *and* Tq *be continual proportionals; and the point* Q *be the focus of the incident rays, the point* q *ſhall be the focus of the reflected ones.*

Let *QA* and *Aq* be an incident and reflected ray (produced) making equal angles with the perpendicular *AE*; and the reflected ray *Aq* (produced) will cut *QE* (produced) ſomewhere in *q*, as being in the plane of incidence *EAQ*[f]. Draw *EG* parallel to *Aq*, and let it meet *AQ* in *G*; and alſo *Eg* parallel to *AQ*, meeting *Aq* in *g*: then becauſe the angles *EAG*, *EAg* are equal[g], it follows that the triangles *EAG*, *EAg* are equiangular at their common baſe *AE*[h], and therefore equicrural[i]; and alſo equal to each other; and conſequently each ſide of the equilateral figure *AGEg*, in its vaniſhing ſtate when *A* comes to *C*, will be equal to half its diagonal *AE*[k], or by conſtruction to *ET*. Now becauſe the triangles *GQE*, *gEq* are equiangular[l], it will be as *GQ* to *GE*, ſo *gE* to *gq*[m]; that is, when the point *A* is coinciding with *C*, and conſequently the points *G*, *g* with *T*, as *TQ* to *TE* ſo *TE* to *Tq*[n]. *Q. E. D.*

208. *Corol.* 1. If *q* be the focus of incident rays, *Q* will be the focus of the reflected ones[o].

209. *Corol.* 2. The rays that belong to *Q* may be reckoned parallel when the diſtance *TQ* is infinite, and then by this propoſition its reciprocal *Tq* becomes nothing; which is the ſecond propoſition.

Fig. 299.

210. *Corol.* 3. Hence alſo we may deduce the firſt propoſition; for ſuppoſing *Q* the focus of incident rays upon the convex ſurface *AB*; ſince *TQ*, *TC*, *Tq* are continual proportionals, it is well known that their differences *CQ*, *Cq* muſt become equal when the lines themſelves are infinite-

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[a] Art. 7.

[b] Art. 8.

[c] Euc. I. 6.

[d] Art. 204.

[e] Art. 11.

[f] Art. 7.

[g] Art. 8.

[h] Euc. I. 29.

[i] Euc. I. 6.

[k] Art. 204.

[l] Euc. I. 29.

[m] Euc. VI. 4.

[n] Art. 204.

[o] Art. 11.

ly great; that is when the ſurface becomes a plane by removing its center to an infinite diſtance.

The figures ſerve for the caſes of a convex ſurface ſuppoſing the incident rays to go backwards in the ſame lines produced through the ſurface.

211. By the demonſtrations of the two laſt propoſitions, it appears that the focus of reflected rays there determined, is nothing elſe in ſtrictneſs of geometry, but the interſection of the axis of the ſurface, that is of the ray paſſing through its center, and of the neareſt rays to it: and alſo that other rays interſect the axis in different points farther and farther from that focus, as they fall farther and farther from the vertex of the ſurface. So that a ſpherical ſurface cannot poſſibly reflect all the incident rays to a ſingle point. Nevertheleſs when theſe aberrations of the remoter rays from the geometrical focus ſhall be conſidered, it will appear hereafter, that the denſity of their interſections, near that focus, is immenſely greater than their denſity at any conſiderable diſtance from it. So that in phyſical things, the focus of all the rays, that fall almoſt perpendicular upon a ſpherical ſurface, may be conſidered as a phyſical point. And the ſame is to be underſtood of the focus of refracted rays, as will appear by the like ſort of demonſtrations.

Fig. 300.

212. Hence it appears that the focus of rays reflected from any curved ſurface whatever, muſt be reckoned the ſame as if they were reflected from a ſpherical ſurface of an equal curvity to that ſurface about the points of incidence. As if *CD* be any curve whatever, *C* the point of incidence, *CEQ* perpendicular to the curve, or to its tangent at *C*, *CE* the radius of a circle *ACB* of the ſame degree of curvity at *C*; the rays coming parallel to *CE*, will be reflected to the ſame focus *T* from either of the ſurfaces; and alſo the rays that flow from any point *Q*, will be reflected by either ſurface to the ſame focus *q*. Becauſe we conſider the focus of thoſe rays only, that fall upon the common points of both curves about *C*, all the reſt being diſperſed much thinner into other places.

213. In all theſe propoſitions when the focuſes *Q*, *q* lye on the ſame ſide of the reflecting ſurface, if the incident rays flow from *Q* the reflected ones will flow towards *q*; and if the incident rays flow towards *Q*, the reflected ones will flow from *q*; and the contrary happens when *Q* and *q* are on contrary ſides of the ſurface. Becauſe the incident and reflected rays go contrary ways.

CHAPTER II.

To determine the place, magnitude and ſituation of images formed by reflected rays.

PROPOSITION I.

214. IMAGES *formed by reflections from a plane ſurface are ſimilar and equal to the objects; and their parts have the ſame ſituation with reſpect to the backſide of the plane as the parts of the object have with reſpect to its foreſide.*

From any number of points P, Q, R of an object in any ſituation, draw the perpendiculars PA, QC, RB to the plane ACB, and produce them through it to the points p, q, r, each as far behind the plane as P, Q, R are before it. The points p, q, r being the reſpective focuſes of the rays that belonged to P, Q, R[a], and being evidently in the ſame order, together with infinite others, will conſtitute an image of the object, equal to it in the whole and in every correſponding part, and alike ſituated: as will appear by conceiving the ſurface of the object, and of its image, divided into correſponding lines, ſuch as PQR, pqr, by planes ſuch as $PprR$ perpendicular to the reflecting plane; and by folding up or doubling each plane in its line of interſection, AB, with the reflecting plane. For the parts of each plane on each ſide of AB will exactly cover each other, as appears by the conſtruction. *Q. E. D.*

Fig. 301, 302.

[a] Art. 292.

PROPOSITION II.

215. *If an arch of a circle* PQR, *concentrick to a concave or convex ſpherical ſurface* AB, *be conſidered as an object, its image* pqr *will alſo be a ſimilar concentrick arch, whoſe length will be to the length of the object, in the ratio of their diſtances from the common center* E; *and its ſituation will be erect or inverted, according as it is on the ſame or the oppoſite ſide of the center to the object.*

Fig. 303 to 306.

For as the focus Q was found by making TQ, TE, Tq continual proportionals in the line QE drawn through the center[b]; ſo the focus p, of rays that belong to any other point P, is found by drawing PEA, and biſecting EA in S, and by making SP, SE, Sp continual proportionals. The two firſt terms of one proportion are ſeverally equal to the two firſt of the other; and conſequently the third terms Tq, Sp are equal; and thence Ep and Eq are equal. The ſame being true of every point of the circular object PQR, ſhews that its image pqr is a concentrick arch, ſimilar to it, both being terminated by the ſame lines EPp, ERr; and conſequently their lengths are in the ſame ratio as their ſemidiameters.

[b] Art. 207.

EQ, Eq. Laſtly according as the correſponding extremities P and p, of the object and image, are on the ſame or oppoſite ſides of the center E, they are alſo on the ſame or oppoſite ſides of their middle points Q, q, that is the image is accordingly erect or inverted. *Q. E. D.*

216. *Corol.* The ſmaller the circular object is with reſpect to its radius or diſtance from the center, the nearer it approaches in ſhape to a ſtraight line, and ſo does its ſimilar image. Conſequently a ſmall ſtraight object, placed at a good diſtance from the center of the glaſs, may be reckoned to have a ſtraight image very nearly: though in ſtrictneſs of geometry it is an arch of a conick ſection, whoſe determination may be ſeen in the remarks.

217. The images of all ſorts of objects may be determined, by finding the images of their out-lines, by the foregoing propoſitions. For inſtance, if the plane of the figures PER, pEr be turned round their common diameter QEq, the circular ſurface generated by pqr will be the image of the circular object generated by PQR: and if the ſame figures PER, pEr be moved a little about an axis EF, ſituated in their own plane, and perpendicular to the diameter QEq, the curvilinear figure generated by this motion of pqr, will be the image of a ſimilar figure generated by PQR. Becauſe the reflecting arch ACB generates the reflecting ſpherical ſurface at the ſame time.

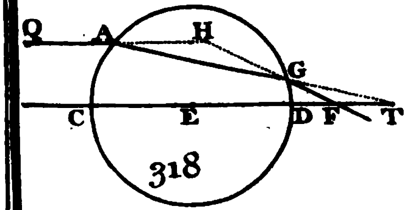
218. But if the whole figure $PERrp$ be moved parallel to itſelf in a direction EF, now perpendicular to its own plane, ſo that the arch ACB may generate a portion of a cylindrical ſurface, the figure deſcribed by this motion of pqr, will ſtill be the image of that deſcribed by PQR; but will not be ſimilar to it, except when they are placed at equal diſtances on each ſide the center E, and conſequently are equal to each other: and their diſſimilitude will be ſo much the greater as the diſproportion between Eq and EQ, or between their lengths pr, PR, is greater; their breadths, deſcribed by the motion aforeſaid, being always equal to each other.

CHAPTER III.

To find the focus of rays falling almoſt perpendicularly upon any refracting ſurface, ſphere or lens.

DEFINITION.

219. THE ſine of an angle ABC, or of an arch AC that meaſures that angle, is a line AD drawn from the extremity of one of
Fig. 307, 308. the ſemidiameters, AB, BC, perpendicular to the other, produced if the angle be obtuſe. And therefore an angle ABC and its complement ABE, to two right angles, have each the ſame ſine AD; and when the ſines of ſeveral



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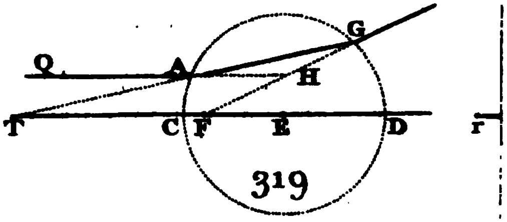

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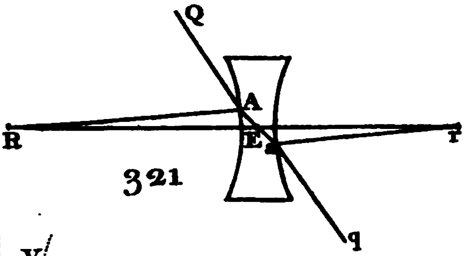

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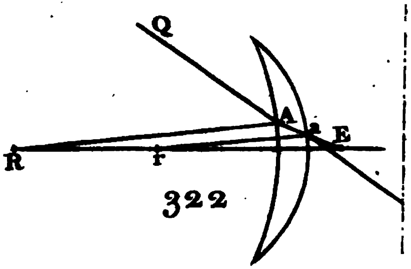

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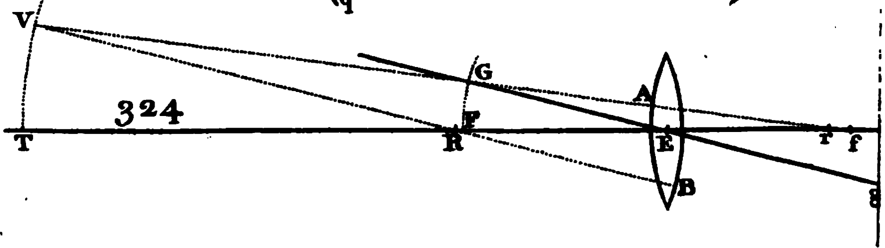

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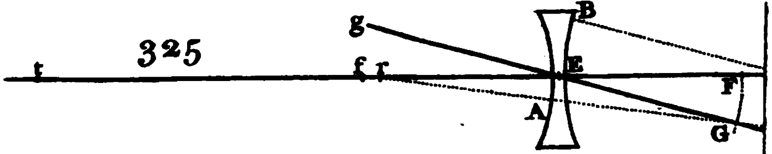

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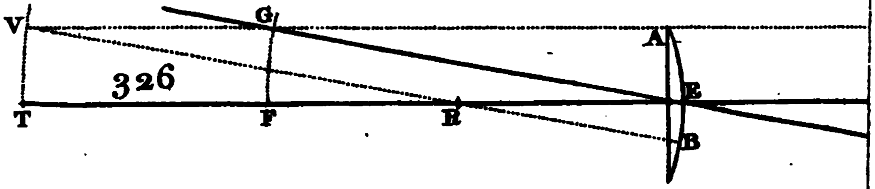

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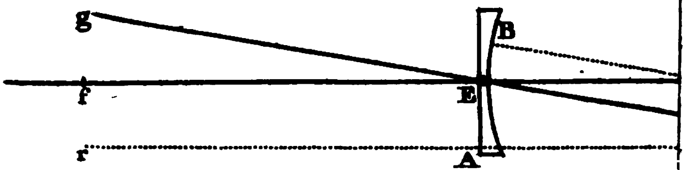

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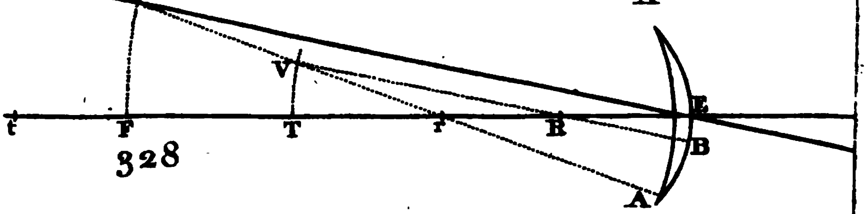

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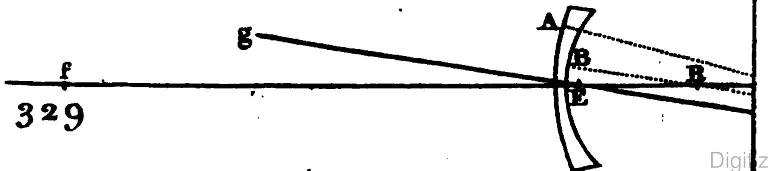

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ſeveral angles are compared together, they are always underſtood to belong to the ſame or to equal circles.

220. The ſines of very ſmall angles, and of their complements, become at laſt inſenſibly different from the arches that meaſure them; and conſequently are proportionable to the angles themſelves.

LEMMA.

Fig. 309.

221. *The ſines of the angles of any triangle are proportionable to the oppoſite ſides: as in the triangle* ABC, *the ſine of the angle* ABC *is to the ſine of the angle* BCA, *as* CA *to* AB.

For the perpendiculars *CD*, *BE* upon thoſe ſides *AB*, *AC* produced, are the ſines of the angles *ABC*, *BCA* or *BCE* with reſpect to circles whoſe radius is *BC*[a]. And ſince the triangles *CAD*, *BAE* are equiangular[b], we have *CD* to *BE* as *CA* to *AB*. *Q.E.D.*

a Art. 219.
b Euc. I. 32. cor. 2.

222. *Corol.* Small angles, as *BAC*, *BCE*, ſubtended by the ſame perpendicular *BE*, are reciprocally as their legs *BA*, *BC* or *EA*, *EC*. For the angle *BAC* is to *BCE*, when very ſmall, as the ſine of *BAC* to the ſine of *BCE*[c], or as *BC* to *BA*[d] or as *EC* to *EA*[e].

c Art. 220.
d Art. 221.
e Art. 204.

PROPOSITION I.

Fig. 310 to 313.

223. *Let* ACB *be a refracting plane, and* Q *the focus of the incident rays, and* QC *a perpendicular to that plane; and if* qC *be taken in this perpendicular, on the ſame ſide of the plane as* QC, *and in proportion to* QC, *as the ſine of incidence to the ſine of refraction, the point* q *ſhall be the focus of the refracted rays.*

For let the lines *QA* and *Aq*, produced as in the figures, repreſent an incident and a refracted ray, cutting *QC* in any point *q* whatever; and ſince a perpendicular to the plane at *A* is parallel to *QC*, the angle *AQC* will be equal to the angle of incidence, and *AqC* to the angle of refraction. Therefore ſince equal angles have equal ſines, the ſine of incidence is to the ſine of refraction (as the ſine of the angle *AQC*, to the ſine of *AqC*, or as *Aq* to *AQ*[f]; that is when the ray *QA* is almoſt perpendicular to the plane *AB*) as *Cq* to *CQ*[g]. *Q.E.D.*

f Art. 221.
g Art. 204.

PROPOSITION II.

Fig. 314. to 317.

224. *Let* ACB *repreſent a refracting ſpherical ſurface whoſe center is* E, *and let the incident rays as* DA *come parallel to any ſemidiameter* CE, *in which produced forward or backward, according as the denſer medium is convex or concave, take* CT *to* CE *as the ſine of incidence to the difference of the ſines, and* T *will be the focus of the refracted rays.*

For let the refracted ray *AT* (produced) cut the ſemidiameter *CE* produced, in any point *T* whatever; and ſince the ſemidiameter *EA* is perpendicular to the refracting ſurface at *A*, the angle of incidence will be equal to the angle *AEC*, and the angle of refraction, or its complement to two right ones, will be EAT; conſequently the ſine of incidence is to the ſine of refraction, (as the ſine of the angle AEC, to the ſine of EAT, or as AT to TE[a], that is, when A comes neareſt to C, and ſo the incident rays are almoſt perpendicular to the ſurface,) as CT to TE[b]; and disjointly[c] the ſine of incidence is to the difference of the ſines as CT to CE. *Q.E.D.*

[a] Art. 221.
[b] Art. 204.
[c] Euc. V. 17.

225. *Corol.* 1. CT is to TE as the ſine of incidence to the ſine of refraction.

226. *Corol.* 2. If this point T be the focus of incident rays, the refracted rays will go parallel to TE[d].

[d] Art. 11.

PROPOSITION III.

Fig. 318, 319.

227. *When parallel rays fall upon a ſphere, either denſer or rarer than the ambient medium, in the diameter* CD *produced, which is parallel to the incident rays, as* QA, *let* T *be their focus after their firſt refraction at* AC; *and the point* F *which biſects* TD *ſhall be their focus after their ſecond refraction at* DG.

For let the incident and emergent rays, QA, FG produced, meet in H, and ſince the refractions at A and G are equal, as appears by ſuppoſing a ray to go both ways along the chord AG, the triangle AHG is equiangular at its baſe AG, and therefore equicrural[e]; and ſo is the ſimilar triangle GFT, the lines AH, FT being parallel. Therefore when A approaches toward C till G is coinciding with D and the triangle GFT is vaniſhing, the leg GF will become equal to half the baſe GT; that is DF will become equal to half DT[f]. *Q.E.D.*

[e] Euc. I. 6.
[f] Art. 204.

LEMMA.

Fig. 320 to 323.

228. *There is a certain point* E *within every double convex or double concave lens, through which every ray that paſſes, will have its incident and emergent parts* QA, aq *parallel to each other: but in a plano-convex or plano-concave lens that point* E *is removed to the vertex of the concave or convex ſurface; and in a meniſcus and in that other concavo-convex lens, it is removed a little way out of them, and lyes next to the ſurface which has the greateſt curvity.*

For let REr be the axis of the lens joining the centers R, r of its ſurfaces A, a. Draw any two of their ſemidiameters RA, ra parallel to each other, and join the points A, a, and the line Aa will cut the axis in the point E above deſcribed. For the triangles REA, rEa being equiangular, RE will be to Er in the given ratio of the ſemidiameters RA, ra; and conſequently the point E is invariable in the ſame lens. Now ſuppoſing a ray to paſs both ways along the line Aa, it being equally inclined to the perpendiculars to the ſurfaces, will be equally bent and contrary ways in going out of the lens; ſo that its emergent parts AQ, aq will be parallel. Now any of theſe lenſes will become plano-convex or plano-concave,

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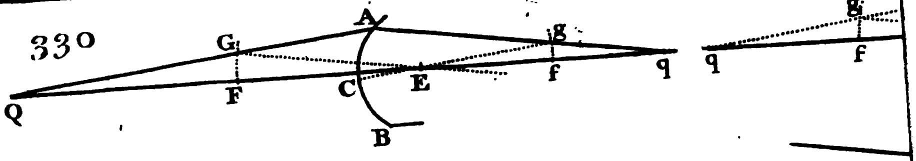

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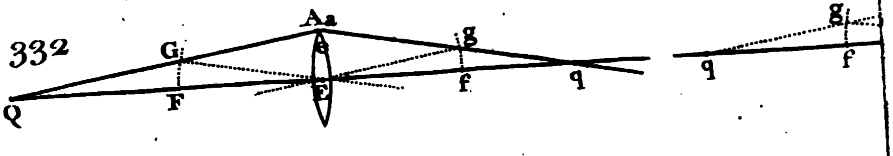

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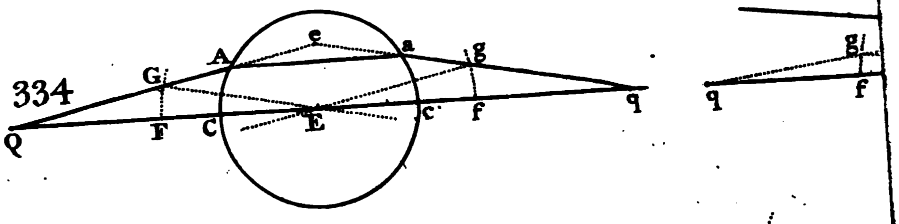

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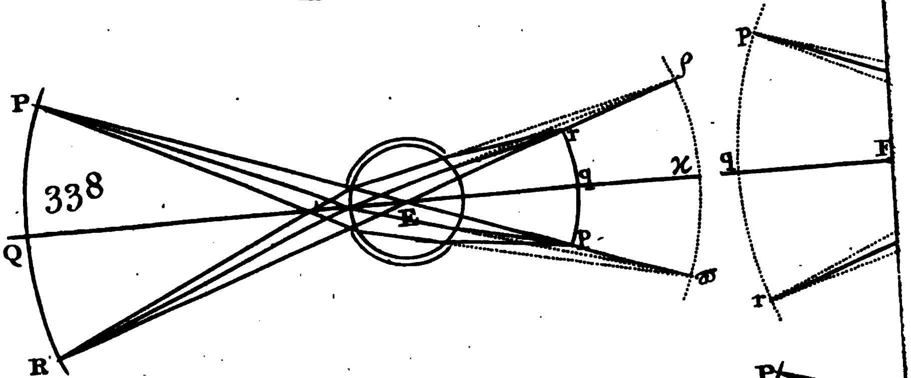

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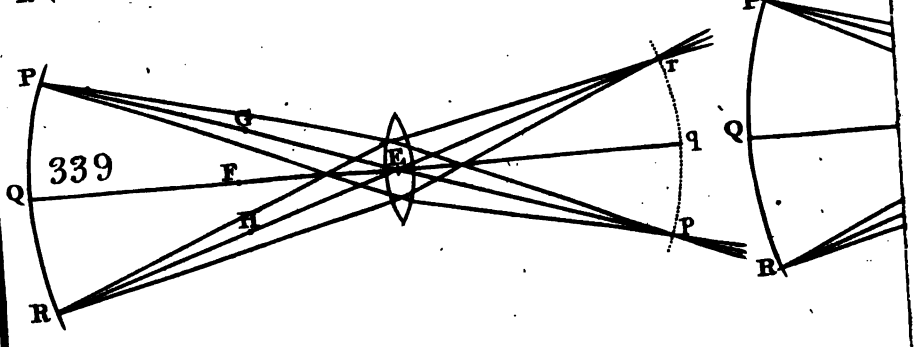

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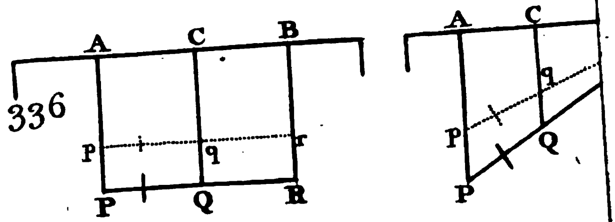

cave, by conceiving one of the ſemidiameters RA, ra to become infinite and conſequently to become parallel to the axis of the lens, and then the other ſemidiameter will coincide with the axis; and ſo the points A, E or a, E will coincide. *Q.E.D.*

229. *Corol.* Hence when a pencil of rays falls almoſt perpendicularly upon any lens, whoſe thickneſs is inconſiderable, the courſe of the ray which paſſes through E, above deſcribed, may be taken for a ſtraight line paſſing through the center of the lens, without ſenſible error in ſenſible things. For it is manifeſt from the length of Aa and from the quantity of the refractions at its extremities, that the perpendicular diſtance of AQ, aq when produced, will be diminiſhed both as the thickneſs of the lens and the obliquity of the ray is diminiſhed.

PROPOSITION IV.

230. *To find the focus of parallel rays falling almoſt perpendicularly upon any given lens.*

Let E be the center of the lens, R and r the centers of its ſurfaces, Rr its axis, gEG a line parallel to the incident rays upon the ſurface B, whoſe center is R. Parallel to gE draw a ſemidiameter BR, in which produced let V be the focus of the rays after their firſt refraction at the ſurface B, and joining Vr let it cut gE produced in G, and G will be the focus of the rays that emerge from the lens.

Fig. 324 to 329.

For ſince V is alſo the focus of the rays incident upon the ſecond ſurface A, the emergent rays muſt have their focus in ſome point of that ray which paſſes ſtraight through this ſurface; that is in the line Vr, drawn through its center r: and ſince the whole courſe of another ray is reckoned a ſtraight line gEG[a], its interſection G with Vr determines the focus of them all. *Q.E.D.*

[a] Art. 229.

231. *Corol.* 1. When the incident rays are parallel to the axis rR, the focal diſtance EF is equal to EG. For let the incident rays that were parallel to gE be gradually more inclined to the axis till they become parallel to it; and their firſt and ſecond focuſes V and G will deſcribe circular arches VT and GF whoſe centers are R and E. For the line RV is invariable; being in proportion to RB in a given ratio of the leſſer of the ſines of incidence and refraction to their difference[b]; conſequently the line EG is alſo invariable, being in proportion to the given line RV in the given ratio of rE to rR, becauſe the triangles EGr, RVr are equiangular.

[b] Art. 224.

232. *Corol.* 2. The laſt proportion gives the following rule for finding the focal diſtance of any thin lens. As Rr, the interval between the centers of the ſurfaces, is to rE, the ſemidiameter of the ſecond ſurface, ſo is RV or RT, the continuation of the firſt ſemidiameter to the firſt focus, to EG or EF, the focal diſtance of the lens. Which according as the lens

is thicker or thinner in the middle than at its edges, muſt lye on the ſame ſide as the emergent rays or the oppoſite ſide.

233. *Corol.* 3. Hence when rays fall parallel on both ſides of any lens the focal diſtances EF, Ef are equal. For let rt be the continuation of the ſemidiameter Er to the firſt focus t of rays falling parallel upon the ſurface A; and the ſame rule that gave rR to rE as RT to EF, gives alſo rR to RE as rt to Ef. Whence Ef and EF are equal, becauſe the rectangles under rE, RT and alſo under RE, rt are equal. For rE is to rt and alſo RE to RT in the ſame given ratio[a].

a Art. 224.

234. *Corol.* 4. Hence in particular in a double-convex or double-concave lens made of glaſs, it is as the ſum of their ſemidiameters (or in a meniſcus as their difference) to either of them, ſo is double the other, to the focal diſtance of the glaſs. For the continuations RT, rt are ſeverally double their ſemidiameters: becauſe in glaſs ET is to TR and alſo Et to tr as 3 to 2[b].

b Art. 225. 13.

235. *Corol.* 5. Hence if the ſemidiameters of the ſurfaces of the glaſs be equal, its focal diſtance is equal to one of them; and is equal to the focal diſtance of a plano-convex or plano-concave glaſs whoſe ſemidiameter is as ſhort again. For conſidering the plane ſurface as having an infinite ſemidiameter, the firſt ratio of the laſt mentioned proportion may be reckoned a ratio of equality.

PROPOSITION V.

Fig. 330 to 335.

236. *The focus of incident rays upon a ſingle ſurface, ſphere or lens being given, it is required to find the focus of the emergent rays.*

Let any point Q be the focus of incident rays upon a ſpherical ſurface, lens, or ſphere, whoſe center is E; and let other rays come parallel to the line QEq the contrary way to the given rays, and after refraction let them belong to a focus F; then taking Ef equal to EF, ſay as QF to FE ſo Ef to fq, and placing fq the contrary way from f to that of FQ from F, the point q will be the focus of the refracted rays, without ſenſible error; provided the point Q be not ſo remote from the axis, nor the ſurfaces ſo broad as to cauſe any of the rays to fall too obliquely upon them.

For with the center E and ſemidiameter EF or Ef deſcribe two arches FG, fg cutting any ray $QAaq$ in G and g, and draw EG and Eg. Then ſuppoſing G to be a focus of incident rays, (as GA,) the emergent rays (as agq) will be parallel to GE[c]; and on the other hand ſuppoſing g another focus of incident rays (as ga,) the emergent rays (as AGQ,) will be parallel to gE. Therefore the triangles QGE, Egq are equiangular, and conſequently QG is to GE as Eg to gq; that is, when the ray $QAaq$ is the neareſt to QEq, QF is to FE as Ef to fg[d]. Now when Q accedes to F and coincides with it, the emergent rays become parallel, that is q recedes

c Art. 225. 231. 233.

d Art. 204.

recedes to an infinite diſtance; and conſequently when *Q* paſſes to the other ſide of *F*, the focus *q* will alſo paſs through an infinite ſpace from one ſide of *f* to the other ſide of it. *Q.E.D.*

237. *Corol.* 1. In refractions at a ſpherical ſurface *AC*, the focus *q* may alſo be found by this rule, as *QF* to *FC* ſo *Cf* to *fq*; becauſe *FC* and *ef* and alſo *FE* and *Cf* are equal[a].

238. *Corol.* 2. It may alſo be found by this rule, as *QF* to *QE* ſo *QC* to *Qq*; placing *Qq* ſo that all the four diſtances from *Q* may lye on one ſide of it, or elſe two on each. For the triangles *QGE*, *QAq* being equiangular we have *QG* to *QE* as *QA* to *Qq*.

239. *Corol.* 3. In a ſphere or lens the focus *q* may be found by this rule, as *QF* to *QE* ſo *QE* to *Qq*, to be placed the ſame way from *Q* as *QF* lyes from *Q*. For let the incident and emergent ray *QA*, *qa* be produced till they meet in *e*; and the triangles *QGE*, *Qeq* being equiangular, we have *QG* to *QE* as *Qe* to *Qq*; and when the angles of theſe triangles are vaniſhing, the point *e* will coincide with *E*; becauſe in the ſphere the triangle *Aea* is equiangular at the baſe *Aa*, and conſequently *Ae* and *ae* will at laſt become ſemidiameters of the ſphere. In a lens the thickneſs *Aa* is inconſiderable.

240. *Corol.* 4. In all caſes the diſtance *fq* varies reciprocally as *FQ* does; and they lye contrary ways from *f* and *F*; becauſe the rectangle or the ſquare under *EF* and *Ef*, the middle terms in the foregoing proportions, is invariable.

241. *Corol.* 5. Convex lenſes of different ſhapes that have equal focal diſtances, when put into each others places, have equal powers upon any pencil of rays to refract them to the ſame focus. Becauſe the rules abovementioned depend only upon the focal diſtance of the lens, and not upon the proportion of the ſemidiameters of its ſurfaces.

242. *Corol.* 6. The rule that was given for a ſphere of an uniform denſity, will ſerve alſo for finding the focus of a pencil of rays refracted through any number of concentrick ſurfaces, which intercede uniform mediums of any different denſities. For when rays come parallel to any line drawn through the common center of theſe mediums, and are refracted through them all, the diſtance of their focus from that center is invariable, as in an uniform ſphere.

243. *Corol.* 7. When the focuſes *Q*, *q* lye on the ſame ſide of the refracting ſurfaces, if the incident rays flow from *Q*, the refracted rays will alſo flow from *q*; and if the incident rays flow towards *Q*, the refracted will alſo flow towards *q*: and the contrary will happen when *Q* and *q* are on contrary ſides of the refracting ſurfaces. Becauſe the rays are continually going forwards.

The 211th and 212th articles are applicable to refractions as well as reflections.

[a] Art. 225.

CHAPTER IV.

To determine the place, magnitude and ſituation of images formed by refracted rays.

PROPOSITION I.

244. IMAGES *formed by refractions at plane ſurfaces are ſimilar to the objects, and are always erect, or in a ſimilar ſituation to the objects, and on the ſame ſide of the planes.*

Fig. 336, 337. Let PQR be an object radiating upon a refracting plane ACB; to which draw the perpendiculars PA, QC, RB, &c. and in theſe perpendiculars take Ap to AP, Cq to CQ, Br to BR in the given ratio of the ſine of incidence to the ſine of refraction[a]; and the focuſes p, q, r, &c. will conſtitute a ſimilar image in a ſimilar ſituation to the object; the parts pq, qr being in the ſame ratio as PQ, QR. This is ſelf-evident when the object is parallel to the refracting plane; and when it is inclined, produce it till it cuts the plane in D; and the produced image will alſo cut it in the ſame point D. For ſuppoſing the perpendicular BrR to move towards D, the lines BR, Br being in a given ratio will vaniſh both together: and becauſe the triangle pDP is cut by parallel lines qQ, rR, it will be as pq to PQ (ſo qD to QD) ſo qr to QR[b]; and alternately pq to qr as PQ to QR. In like manner if the rays that belong to the focuſes p, q, r be refracted again by another plane, either parallel or inclined to AB, their ſecond focuſes will conſtitute a ſecond image, ſimilar to the firſt image and conſequently ſimilar to the object; and ſo on. *Q. E. D.*

PROPOSITION II.

Fig. 338 to 341. 245. *If an arch of a circle* PQR *deſcribed upon the center* E, *of a ſpherical ſurface, ſphere or lens, be conſidered as an object, its image* pqr *will be a ſimilar concentrick arch; whoſe length will be to the length of the object in the ratio of their diſtances from the common center* E; *and the image will be erect or inverted, with reſpect to the object, according as they lye on the ſame ſide of the center or on contrary ſides.*

The propoſition is evident by inſpection of the 338th figure in all caſes of refractions made by concentrick ſurfaces; becauſe the parts of theſe ſurfaces are alike expoſed to the parts of the concentrick object. And in a lens the focuſes of all the pencils of parallel rays lye alſo in a concentrick arch GFH; whence Pp and Qq being third proportionals to two pair of equal diſtances PG and PE, QF and QE[c], are alſo equal; and ſo the image pqr is alſo a concentrick arch. Now ſince the axes of

the

[a] Art. 223.

[b] Euc. VI. 2.

[c] Art. 239.

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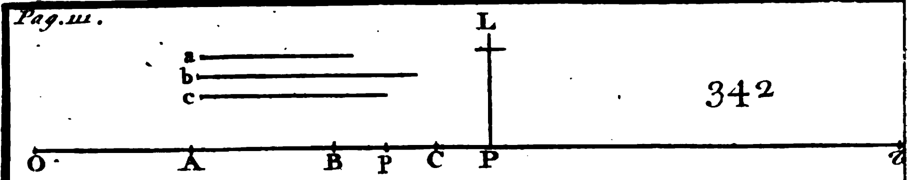

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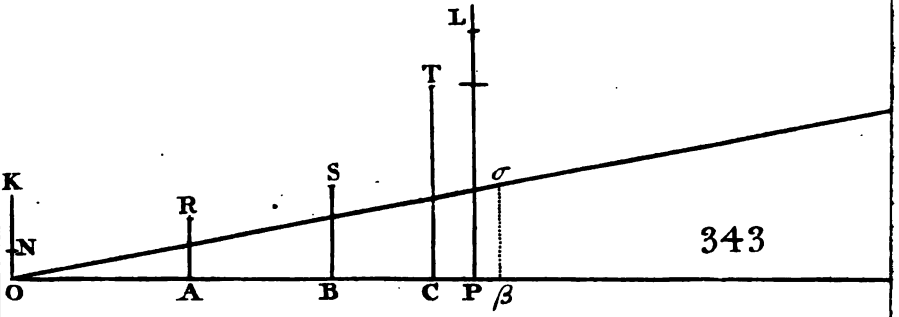

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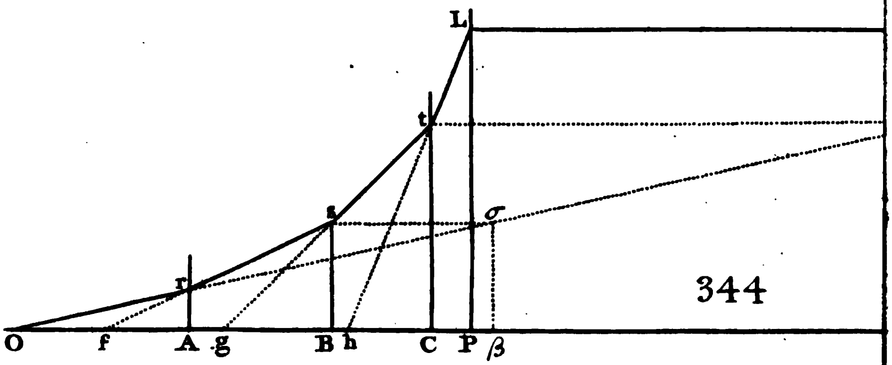

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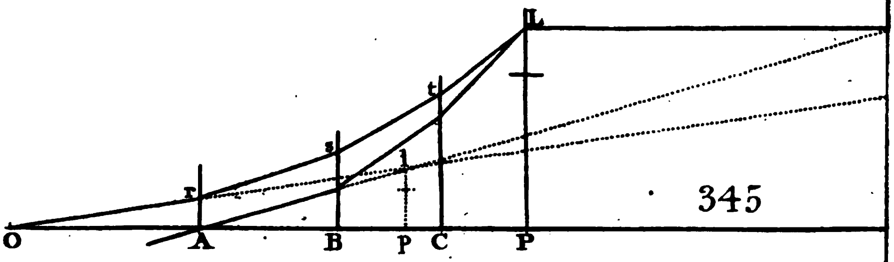

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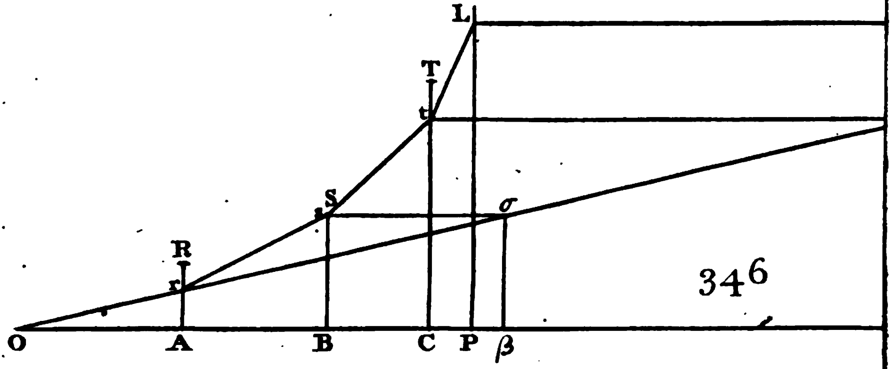

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the pencils are reckoned ſtraight lines paſſing through *E*[a], the angles *pEr*, *PER* are equal; and therefore the ratio of the image to the object, is the ſame as of their diſtances from the center *E*. And according as their correſponding extremities *P*, *p* are on the ſame or on contrary ſides of *E*, they lye on the ſame or contrary ſides of their middle points *Q*, *q*; that is the image is accordingly erect or inverted. *Q. E. D.* [a] Art. 229.

246. *Corol.* The ſmaller the circular object is with reſpect to its radius or diſtance from the center *E*, the nearer it approaches in ſhape to a ſtraight line, and ſo does its ſimilar image. Conſequently a ſmall ſtraight object placed at a good diſtance from the center of the glaſs may be reckoned to have a ſtraight image very nearly[b]: though in ſtrictneſs of geometry it is an arch of a conick ſection determined in the Remarks. And by theſe propoſitions the images of all objects may be determined, by finding the images of their out-lines. Here I conclude the elementary propoſitions, which are alſo demonſtrated algebraically in the Remarks. [b] Art. 204.

CHAPTER V.

To determine the apparent diſtance, magnitude, ſituation, degree of diſtinctneſs and brightneſs, the greateſt angle of viſion and viſible area, of an object ſeen by rays ſucceſſively reflected from any number of plane or ſpherical ſurfaces; or ſucceſſively refracted through any number of lenſes of any ſort, or through any number of different mediums whoſe ſurfaces are plane or ſpherical. With an application to Teleſcopes and Microſcopes.

PROPOSITION I.

247. HAVING *the focal diſtances and apertures of any number of lenſes of any ſort, placed at any given diſtances from one another and from the eye and object, it is required to find the apparent diſtance, magnitude, ſituation, degree of diſtinctneſs and brightneſs of the object ſeen through all the lenſes; together with the greateſt angle of viſion and viſible area of the object, and the particular aperture which limits them both.*

248. Let *PL* be an object viewed by the eye at *O* through any number of lenſes placed at *A*, *B*, *C*, whoſe focal diſtances are the lines *a*, *b*, *c*, and whoſe common axis is the line *OABCP*. The diſtance *OP* may be conſidered as divided by the glaſſes at *A*, *B*, *C*, into two ſuch parts as *OA*, *AP*; *OB*, *BP*; *OC*, *CP*; or into three ſuch parts as *OA*, *AB*, *BP*; *OA*, *AC*, *CP*; or into four ſuch parts as *OA*, *AB*, *BC*, *CP*; and Ap. diſtance. Fig. 342.

ſo on as far as the number of glaſſes permits. All the ſeveral products of ſuch correſponding parts, applyed reſpectively to the focal diſtance, or to the product of the focal diſtances of the glaſſes, which are placed at the point or points of diviſion, will give ſo many ſeveral lines; which muſt be looked upon as negative, if there be an odd number of convex glaſſes at the points of diviſion, otherwiſe as affirmative. Let $O\Pi$ be the ſum of OP and of thoſe ſeveral lines according to their ſigns, and it will be the apparent diſtance of the object.

Ap. magnitude.

249. And its apparent magnitude will be to its true[a] magnitude, as OP to $O\Pi$.

[a] Art. 100.

Ap. ſituation.

250. And if the value of $O\Pi$ be affirmative, the object will appear upright, otherwiſe inverted. Or to expreſs the ſame things in other words, the object will appear through all the glaſſes at the ſame diſtance, of the ſame magnitude, and in the ſame ſituation, as it would appear to the naked eye viewing it from the diſtance $O\Pi$, ſuppoſing it placed upright at Π when $O\Pi$ is affirmative, and inverted when $O\Pi$ is negative.

Ap. diſtinctneſs.

251. When $O\Pi$ is affirmative place it before the eye, otherwiſe behind it; then let the eye be removed from O to A, that its diſtance from the next glaſs may vaniſh; and here let $A\varpi$ be the apparent diſtance of the object PL, to be found and placed by the ſame rules as $O\Pi$ was. Then let Ap be to $A\varpi$ as AO to the difference of $O\Pi$ and $A\varpi$, if they lye the ſame way from O and A, otherwiſe to their ſum; and let the order of the points A, p, ϖ be the ſame as the order of the points O, p, Π; and from the ſituation of this point p, a judgment may be formed of the degree of diſtinctneſs with which the object will appear. Becauſe the rays flowing from P, by paſſing through the glaſſes, will be diſpoſed to fall upon the eye, in the ſame manner as if the glaſſes being removed, they tended from the point p, when it falls before the eye, or towards the point p, when it falls behind the eye.

Viſual angle. Fig. 343.

252. Let the lines AR, BS, CT be the ſemidiameters of the given apertures of the glaſſes A, B, C; and let $O\beta$ be the apparent diſtance of the line BS, ſeen through the glaſs A; and $O\gamma$ the apparent diſtance of the line CT ſeen through the glaſſes A and B; to be found as above. Erect the perpendiculars $\beta\sigma$ equal to BS, and $\gamma\tau$ equal to CT, and then the leaſt of the angles, which any one of the perpendiculars AR, $\beta\sigma$, $\gamma\tau$, ſubtends at O, will be half the greateſt angle of viſion.

Viſible area.

253. Let this angle be $\beta O\sigma$, and let $O\sigma$, produced, cut a perpendicular to the axis at Π in Λ; then PL taken equal to $\Pi\Lambda$ will be the ſemidiameter of the greateſt area of the object PL, that can be ſeen at one view from O, through the given apertures of all the glaſſes: and therefore PL or $\Pi\Lambda$, the ſemidiameter of the viſible area, will be to $\beta\sigma$ or BS, the ſemidiameter of the aperture which limits it, as $O\Pi$ the apparent diſtance of the area, to $O\beta$ the apparent diſtance of that aperture.

Where limited.

254. And by the ſuppoſition that $\beta O\sigma$ is the leaſt of all the angles ſubtended at O by any of the given lines AR, $\beta\sigma$, $\gamma\tau$, it will follow that the aperture which limits the viſual angle and viſible area, belongs to the glaſs B.

Ap. brightneſs.

255. Since the magnitude of the pupil is ſubject to be varied by various degrees of light, let NO be its ſemidiameter when the object PL is viewed by the naked eye from the diſtance OP; and upon a plane that touches the eye at O let OK be the ſemidiameter of the greateſt area, viſible through all the glaſſes to another eye at P, to be found as PL was; or which is the ſame thing let OK be the ſemidiameter of the greateſt area inlightened by a pencil of rays flowing from P through all the glaſſes; and when this area is not leſs than the area of the pupil, the point P will appear juſt as bright through all the glaſſes, as it would do if they were removed: but if the inlightened area be leſs than the area of the pupil, the point P will appear leſs bright through the glaſſes, than if they were removed, in the ſame proportion as the inlightened area is leſs than the pupil. And theſe proportions of apparent brightneſs would be accurate, if all the incident rays were tranſmitted through the glaſſes to the eye, or if only an inſenſible part of them were ſtopt.

DEMONSTRATION.

Ap. diſtance. Fig. 344.

256. For let any ray $OrstL$, of a pencil ſuppoſed to flow from the eye to the object, belong ſucceſſively to the focuſes f, g, h, after its ſucceſſive refractions at the glaſſes Ar, Bs, Ct; and then let it fall upon the object at L. The point L will therefore be ſeen by the ray $LtsrO$ returning back along the ſame lines to the eye at O[a]. Let $L\Lambda$ drawn parallel to OP meet the viſual ray Or, produced, in Λ; and let the rectangle $PL\Lambda\Pi$ be compleated; and $O\Pi$ will be the apparent diſtance of the object[b]. Let us at firſt ſuppoſe all the lenſes to be concaves; and ſince the triangles OAr, $O\Pi\Lambda$ are equiangular, we have OA to $O\Pi$ in the ſame ratio as Ar to $\Pi\Lambda$ or PL; or in a ratio compounded of Ar to Bs, Bs to Ct, Ct to PL; or compounded of fA to $fA+AB$, gB to $gB+BC$, hC to $hC+CP$; and conſequently $O\Pi = OA\times\frac{fA+AB}{fA}\times\frac{gB+BC}{gB}\times\frac{hC+CP}{hC}$; by which theorem the apparent diſtance $O\Pi$ will be given ſo ſoon as fA, gB, hC can be found. Theſe may be found by Art 239 as follows; $fA=\frac{OA\times a}{OA+a}$; $gB=\frac{\overline{fA+AB}\times b}{fA+AB+b}$; $hC=\frac{\overline{gB+BC}\times c}{gB+BC+c}$. Hence it is eaſy to collect, that if the eye at O views an object at B through one glaſs at A, its apparent diſtance $O\beta=OB+\frac{OAB}{a}$; that if the eye at O

[a] Art. 11.

[b] Art. 139.

views an object at C, through two glaſſes at A, B, its apparent diſtance $O\gamma = OC + \frac{OAC}{a} + \frac{OBC}{b} + \frac{OABC}{ab}$; that if the eye at O views an object at P through three glaſſes at A, B, C, its apparent diſtance $O\Pi = OP + \frac{OAP}{a} + \frac{OBP}{b} + \frac{OCP}{c} + \frac{OABP}{ab} + \frac{OACP}{ac} + \frac{OBCP}{bc} + \frac{OABCP}{abc}$; and ſo on continually as the ſolution of the problem directs. Now if any of the lenſes be convex, the focal diſtances of ſuch lenſes muſt be looked upon as negative, ſince they lye the contrary way to thoſe of concave lenſes, when the incident rays come the ſame way upon both ſorts: therefore the terms which involve any odd number of convex lenſes at the points of diviſion, muſt be looked upon as negative

Ap. magnitude.

The determination of the apparent magnitude is evident by Art. 141.

Ap. ſituation.

And that of the apparent ſituation by the latter part of Art. 139.

Ap. diſtinctneſs. Fig. 345.

257. Compleat the rectangle $LP\varpi\lambda$ and join $A\lambda$ meeting $O\Lambda$ in l; and the line lp, drawn perpendicular to the axis of the glaſſes, will be the laſt image of the object LP. Becauſe the ſame point L is ſeen by a ray which falls upon the eye at O in the direction ΛO, and alſo by a ray which falls upon the eye at A in the direction λA; and therefore the point l where theſe directions croſs one another is the focus of the emergent rays. Now ſince the triangles Apl, $A\varpi\lambda$ and alſo Opl, $O\Pi\Lambda$ are equiangular, we have Ap to $A\varpi$ as (pl to $\varpi\lambda$ or $\Pi\Lambda$, or as) Op to $O\Pi$, or as $Op \mp Ap$ or OA to $O\Pi \mp A\varpi$, according as p falls without or within the line OA, and conſequently according as $O\Pi$ and $A\varpi$ lye the ſame or contrary ways from O and A. And the order of the poins A, p, ϖ is the ſame as the order of the points A, l, λ, or of the points O, l, Λ, or of the points O, p, Π.

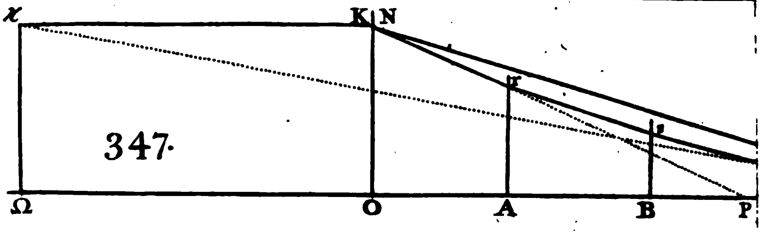
Viſual angle. Fig. 343, 346.

258. Let $O\sigma$ cut the perpendiculars AR in r, $\gamma\tau$ in θ, $\Pi\Lambda$ in Λ, and compleat the rectangles $B\beta\sigma s$, $C\gamma\theta t$, $P\Pi\Lambda L$; then by the ſuppoſition that the angle $\beta O\sigma$ is the leaſt of the angles, which any one of the perpendiculars AR, $\beta\sigma$, $\gamma\tau$ ſubtends at O, it follows that Ar is leſs than AR; and $\gamma\theta$ leſs than $\gamma\tau$, or Ct leſs than CT. But joining rs, st, tL, theſe lines will be deſcribed by a ray paſſing from O to L. Becauſe the lines Or, $Or\sigma$, $Or\theta$, $Or\Lambda$ are the ſeveral apparent diſtances of the points r, s, t, L ſeen in one common direction Or. But in the conſtruction $\beta\sigma$ was taken equal to BS; and ſuppoſing the viſual angle $\beta O\sigma$ to be increaſed never ſo little, the equal perpendiculars $\beta\sigma$, Bs would alſo be increaſed; and then Bs would become bigger than BS, and conſequently the outmoſt ray Lts would be ſtopt at s for want of a larger aperture than BS.

Viſible area.

259. The perpendicular $\Pi\Lambda$ or PL would alſo be increaſed by increaſing the angle $\beta O\sigma$, but this being impoſſible without increaſing BS,

it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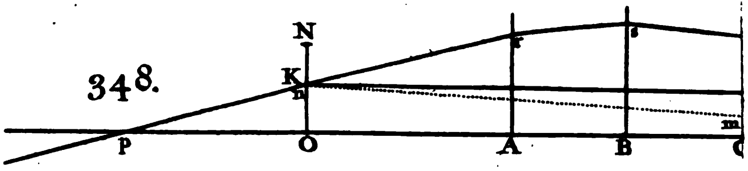
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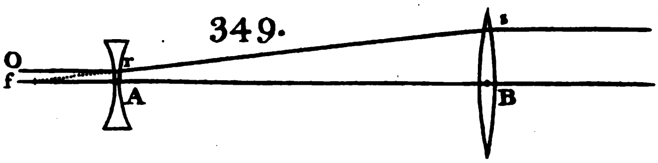
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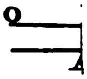

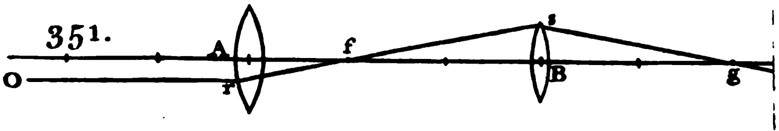
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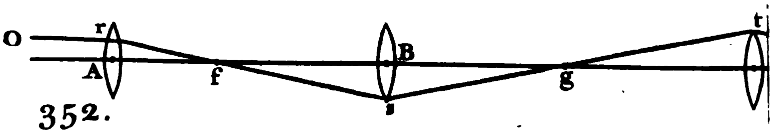
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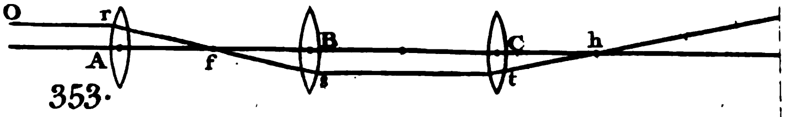
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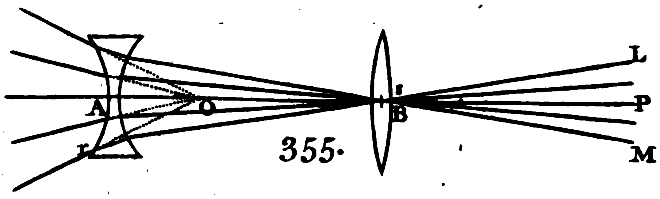
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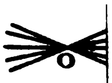

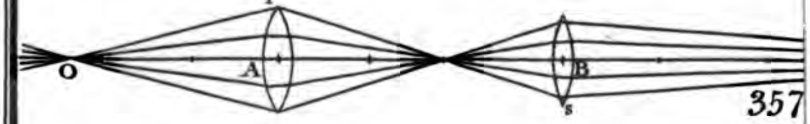
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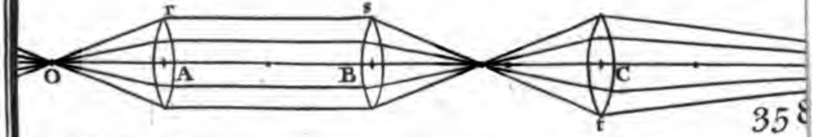
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it is plain that PL is the ſemidiameter of the greateſt area that can be ſeen at one view from O through all the given apertures.

Where limited. 260. And ſo it is manifeſt that the viſion is confined by that aperture at B, which appears through the intermediate glaſſes, under a leſſer angle $\beta O\sigma$, than any other aperture would appear under, if the reſt of the apertures were ſufficiently inlarged for the purpoſe of ſeeing it.

Ap. brightneſs. Fig. 347. 261. If OK be not leſs than ON, the area of the pupil will be totally inlightened by the pencil that flows from P. Let $PtsrN$ be a ray of that pencil, cutting the object-glaſs Ct in t; and ſuppoſing the glaſſes were removed, let an unrefracted ray PMN cut the line Ct in M. Then the quantity of refracted rays which fall upon the line NO, is to the quantity of unrefracted rays which would fall upon it, as the angle CPt to the angle CPM, that is as the apparent magnitude of the line NO, ſeen from P, to the true. And therefore by turning the figure round about the axis OP, the quantity of refracted rays which fill the pupil, is to the quantity of unrefracted rays which would fill it (as the apparent magnitude of any ſurface at O ſeen from P, to the true; or as the apparent magnitude of any ſurface at P ſeen from O to the true[a]; and conſequently) as the apparent magnitude of the leaſt ſurface or phyſical point P, to the true; that is as the picture of the point P formed upon the retina by thoſe refracted rays, to its picture formed by the unrefracted rays. Theſe pictures of the point P are therefore equally bright and cauſe the appearance of P to be equally bright in both caſes. Now let the pupil be larger than the greateſt area inlightened at O by the pencil that flows from P; and ſuppoſing a ſmaller pupil equal to this area, we have ſhewn that the pictures of P upon the retina made by refracted and unrefracted rays would be equally bright; and conſequently each of them would be leſs bright than when the larger pupil is filled with unrefracted rays, in the ſame proportion as the ſmaller pupil, or area inlightened by the refracted rays, is leſs than the larger pupil, inlightened by unrefracted rays. Hitherto I have ſuppoſed the picture of the point P to be diſtinct upon the retina, or proportionable to the angle which meaſures the apparent magnitude of P; let us now ſuppoſe it to be confuſed and the concluſion will not be altered. For the confuſed picture of P diffuſes an equal portion of its own light upon every point (equidiſtant from its center,) quite round its diſtinct bounds, and receives from every point as many ſuch portions of their lights, which came from other points of the object. Fig. 348.

262. *Corol.* 1. While the glaſſes are fixt, if the eye and object be ſuppoſed to change places, the apparent diſtance, magnitude and ſituation of the object will be the ſame as before. For the interval OP being the ſame, and being divided by the ſame glaſſes into the ſame parts, will give the ſame theorem for the apparent diſtance as before[b]; namely PO

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[a] Art. 262.

[b] Art. 248.

$$+\frac{PCO}{c}+\frac{PBO}{b}+\frac{PAO}{a}+\frac{PCBO}{cb}+\frac{PCAO}{ca}+\frac{PBAO}{ba}+\frac{PCBAO}{cba}.$$

Fig. 347, 348. 263. *Corol.* 2. When an object PL is ſeen through any number of glaſſes, the breadth of the principal pencil where it falls on the eye at O, is to its breadth at the object-glaſs C, as the apparent diſtance of the object, to its real diſtance from the object-glaſs; and conſequently in Teleſcopes, as the true magnitude of the object, to the apparent; that is OK is to Ct as $O\Pi$ to PC. For let $K\kappa$ drawn parallel to the axis OP, cut Pt produced in κ, and compleat the rectangle $\kappa KO\Omega$, then $P\Omega$ is the apparent diſtance of an object OK ſeen from P through all the glaſſes; and the triangles $P\Omega\kappa$, PCt being equiangular, we have OK or $\Omega\kappa$ to Ct as $P\Omega$ to PC, or as $O\Pi$ to PC by the foregoing corollary.

264. *Corol.* 3. When the rays flowing from P through all the glaſſes, fall perpendicularly upon a fixt plane at O, their denſity will be uniform in every part of the inlightened area. For ſuppoſing all the incident rays to be tranſmitted, their quantity in the areas at C and at O will be the ſame; and this quantity being uniformly denſe in the area at C, is as this area, or as the area at O (the ratio of theſe areas being invariable[a]) and conſequently is uniformly denſe in the area at O. And whatever part of the light may not be tranſmitted to the area at O; yet equal portions of it will be reflected back from equal portions of any of the ſurfaces of the glaſſes very nearly; (becauſe all the rays fall almoſt perpendicularly upon every ſurface;) and conſequently equal portions of the rays will be intercepted from falling upon equal portions of the area at O.

265. *Corol.* 4. This uniform denſity of the refracted rays in the area at O, is to the uniform denſity of unrefracted rays, which would fall upon it, if the glaſſes were removed and which come from the ſame point P, as the apparent magnitude of the point P, or of any ſurface at P, to the true; ſuppoſing all the rays were tranſmitted. This will appear from the firſt part of the 261ſt article.

266. *Corol.* 5. If the quantity of incident light, in paſſing through the glaſſes, be diminiſhed in no greater a proportion than that of the greateſt aperture of the pupil to the given aperture ON, and the inlightened area be not leſs than the greateſt aperture; the pupil will dilate it ſelf till it receives the ſame quantity of light as in viſion with the naked eye[b]. But when the inlightened area is leſs than the given aperture of the pupil, the natural brightneſs of the object will appear diminiſhed in the glaſſes, in a ratio compounded of the ratio of the aperture of the pupil, to the inlightened area[c], and of the quantity of incident light to the quantity of emergent light.

267. *Corol.* 6. It is evident that an object ſeen through glaſſes may appear as bright as to the naked eye; but never brighter, even though all the incident light be tranſmitted through the glaſſes.

268.

[a] Art. 263.

[b] Art. 264.

[c] Art. 255.

268. *Corol.* 7. The glaſſes and object being fixt, the apparent brightneſs of the point P, ſeen by refracted rays, is invariable wherever the eye is placed, while the pupil is filled with rays that come from that point; but where it is not filled, the apparent brightneſs varies directly in a duplicate ratio of, Op, the diſtance of the pupil from the laſt image of P. For the denſity of the rays, and the apparent magnitude of the point P, and conſequently the magnitude of its picture upon the retina, do all vary reciprocally in a duplicate ratio of Op[a]. Conſequently while the pupil does not vary and is filled with rays, the quantity that enters it is as the area of the picture of P upon the retina; whoſe brightneſs is therefore invariable: but where the pupil is not filled with rays, the quantity that enters the pupil is invariable, while the area of the picture varies reciprocally in a duplicate ratio of Op, and conſequently while its brightneſs varies directly in the duplicate ratio of Op. And this is ſo whatever part of the incident light be ſtopt by the glaſſes. Fig. 348.

269. *Corol.* 8. When the object is ſo remote that the diſtances OP, AP, BP, CP, may be conſidered as equal to one another, then the apparent diſtance $O\Pi = \Theta P$ into $1 + \frac{OA}{a} + \frac{OB}{b} + \frac{OC}{c} + \frac{OAB}{ab} + \frac{OAC}{ac} + \frac{OBC}{bc} + \frac{OABC}{abc}$.

270. *Corol.* 9. Hence when O and b are conjugate focuſes of a pencil of rays refracted through any number of lenſes A, B, C, the ratio of the angles AOr, Cbt, made by the incident and emergent parts of any ray with the axis of the glaſſes, is the ſame as of 1 to $1 + \frac{OA}{a} + \frac{OB}{b} + \frac{OC}{c} + \frac{OAB}{ab} + \frac{OAC}{ac} + \frac{OBC}{bc} + \frac{OABC}{abc}$. For this latter ratio is the ſame as the ratio of OP to $O\Pi$ by cor. 8, that is of the apparent magnitude of a remote object ſeen from O, to its true magnitude ſeen from O or b by the naked eye, or as the angle at O to the angle at b. Fig. 344.

271. *Corol.* 10. Hence if O be the focus of incident rays, the focus of the emergent rays from the laſt glaſs C, may be found by taking Cb to $O\gamma$ or $OC + \frac{OAC}{a} + \frac{OBC}{b} + \frac{OABC}{ab}$ as 1 to $1 + \frac{OA}{a} + \frac{OB}{b} + \frac{OC}{c} + \frac{OAB}{ab} + \frac{OAC}{ac} + \frac{OBC}{bc} + \frac{OABC}{abc}$, and by placing Cb contrary to the courſe of the rays if the ſecond and third terms of this proportion have like ſigns, otherwiſe according to their courſe. For compleating the rectangle $\gamma C t \theta$, Cb is to $O\gamma$ as the angle $\gamma O \theta$ to the angle Cbt[b], that is in the ratio abovementioned, by cor. 9.

272. *Corol.* 11. When the focal diſtances of the glaſſes, and the diſtances of the glaſſes from one another and from the object are ſuch that 1

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[a] Art. 58. 106. 111.

[b] Art. 60.

$+\frac{AP}{a}+\frac{BP}{b}+\frac{CP}{c}+\frac{ABP}{ab}+\frac{ACP}{ac}+\frac{BCP}{bc}+\frac{ABCP}{abc}=0$, the rays of any pencil will fall upon the eye in parallel lines; and then the apparent diſtance $O\Pi$ will be equal to $A\varpi=AP+\frac{ABP}{b}+\frac{ACP}{c}+\frac{ABCP}{bc}$ or $=-a$ into $1+\frac{BP}{b}+\frac{CP}{c}+\frac{BCP}{bc}$: And this apparent diſtance being invariable, the apparent magnitude, ſituation, and degree of diſtinctneſs and brightneſs will alſo be invariable wherever the eye is placed. For the rays flowing from L will fall parallel upon the eye when Ol and Al, or $O\Lambda$ and $A\lambda$ are parallel; and conſequently when $O\Pi=A\varpi$, or $O\Pi-A\varpi=0$. Now by putting $OA=0$ in the value of $O\Pi=OP+\frac{OAP}{a}+\frac{OBP}{b}+\frac{OCP}{c}+\frac{OABP}{ab}+\frac{OACP}{ac}+\frac{OBCP}{bc}+\frac{OABCP}{abc}$, we have $A\varpi=AP+\frac{ABP}{b}+\frac{ACP}{c}+\frac{ABCP}{bc}$; which being taken from $O\Pi$, there remains $O\varpi-A\varpi=1+\frac{AP}{a}+\frac{BP}{b}+\frac{CP}{c}+\frac{ABP}{ab}+\frac{ACP}{ac}+\frac{BCP}{bc}+\frac{ABCP}{abc}=0$; which gives $-a$ into $1+\frac{BP}{b}+\frac{CP}{c}+\frac{BCP}{bc}=AP+\frac{ABP}{b}+\frac{ACP}{c}+\frac{ABCP}{bc}=A\varpi=O\Pi$. Fig. 345.

273. *Corol.* 12. Hence when the object is ſo remote, that the diſtances OP, AP, BP, CP may be conſidered as equal to one another, the rays which fall parallel upon the firſt glaſs will emerge parallel from the laſt, if the glaſſes be ſo diſpoſed that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{AB}{ab}+\frac{AC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=0$; and on the contrary: and then the apparent diſtance $O\Pi=$ ($A\varpi=$) OP into $1+\frac{AB}{b}+\frac{AC}{c}+\frac{ABC}{bc}$ or $=-a\times OP$ into $\frac{1}{b}+\frac{1}{c}+\frac{BC}{bc}$. And conſequently in two concave lenſes A, B, the apparent magnitude is to the true, or OP to $O\Pi$ as $-\frac{1}{a}$ to $\frac{1}{b}$; in three concave lenſes A, B, C, as $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+\frac{BC}{bc}$; and in four concave lenſes A, B, C, D, as $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{BC}{bc}+\frac{BD}{bd}+\frac{CD}{cd}+\frac{BCD}{bcd}$. The unit is rejected in theſe deductions from cor. 11. as being inconſiderable in compariſon to the diſtance of the object.

274. *Corol.* 13. Since the eye, the glaſſes and object are placed in a given order, their intervals OA, AB, BC, &c. muſt be conſidered as being all affirmative, and ſince every term in this equation, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

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$+\frac{AB}{ab}+\frac{AC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=0$, for placing three glaſſes A, B, C, as abovementioned, is affirmative; the ſum of the terms cannot be nothing, and conſequently the rays cannot fall upon the eye in parallel lines, unleſs ſome one of the focal diſtances be negative, or ſome one of the glaſſes be convex. Now in a Teleſcope ſuppoſed to conſiſt of two concave lenſes A, B, we have $\frac{1}{a}+\frac{1}{b}+\frac{AB}{ab}=0$, or $AB=-a-b$; which ſhews that AB the interval of the two glaſſes, muſt be equal to the ſum or difference of their focal diſtances, according as both are convex, or one convex and the other concave. In the firſt caſe we have OP to $O\Pi$ as $\frac{1}{a}$ to Fig. 349, 350.
$-\frac{1}{b}$ or as b to $-a$, by cor. 12, where the negative value of $O\Pi$ ſhews that the object will appear inverted [a]. In the ſecond caſe we have OP to $O\Pi$ a Art. 250.
as b to a, which ſhews that the object will appear upright.

275. *Corol.* 14. For placing three concave glaſſes as above, we had $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{AB}{ab}+\frac{AC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=0$ *, or $b+a+\frac{ab}{c}+$ * Art. 273.
$AB+\overline{AB+BC}\times\frac{b}{c}+\frac{aBC}{c}+\frac{ABC}{c}=0$, or $\overline{AB+a+b},c+\overline{AB+a},b$
$+\overline{AB+a+b}, BC=0$, or $\overline{AB+a+b}, \overline{BC+c}+\overline{AB+a},b$
$=0$. Let all the glaſſes be convex then $AB-a-b$ is to $AB-a$ as b to Fig. 351.
$BC-c$; by which proportion either of the intervals AB, BC will be given when the other is aſſumed as moſt convenient. We had OP to
$O\Pi$ as $\frac{bc}{a}$ to $BC-b-c$ *; where if BC be affirmative or bigger than * Art. 273.
$b+c$ the object will appear upright [b]. Put $BC-b-c=b$, or $BC=2b$ b Art. 250.
$+c$, and the object will appear upright and magnified in the ratio of OP to $O\Pi$ or of c to a, whatever be the length of b. To determine the other interval AB, we found $AB-a-b$ to $AB-a$ as b to $BC-c$ or $2b$ by the aſſumption; conſequently $2AB-2a-2b=AB-a$, and $AB=a+2b$. Hence if we put $b=a$, then $AB=3a$, and $BC=2a+c$.

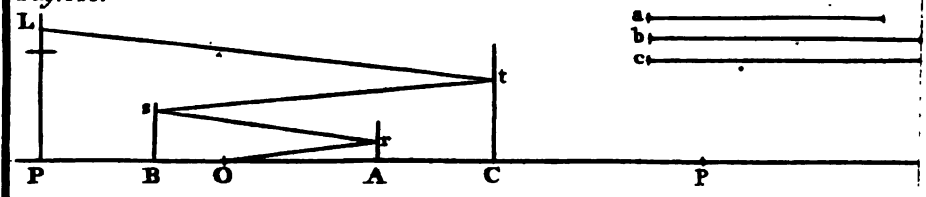
276. *Corol.* 15. But for the rays of a pencil to emerge parallel to one another from any number of glaſſes, it is only neceſſary that their laſt focus ſhould coincide with the principal focus of the laſt glaſs; as is evident by conceiving the emergent rays to return backwards in the ſame parallel lines. Therefore all the intervals of the glaſſes but the laſt may be aſſumed as ſhall be found moſt convenient for other purpoſes. And then if any Fig. 352, 353.
point O be the focus of incident rays their ſucceſſive focuſes f, g, h, i, &c. after their refractions at A, B, C, D, &c. may be eaſily found by theſe
rules [c]; $fA=\frac{OA,a}{OA+a}$; $gB=\frac{\overline{fA+AB},b}{fA+AB+b}$; $hC=\frac{\overline{gB+BC},c}{gB+BC+c}$; $iD=\frac{\overline{hC+CD},d}{hC+CD+d}$; c Art. 256.
&c.

&c. taking due care to obſerve the ſigns of fA, gB, hC, &c. and to place them forwards or according to the courſe of the rays when negative, and backwards when affirmative. For example in a Teleſcope conſiſting of four convex glaſſes, ſuppoſing rays to come parallel upon the eye-glaſs A, the line AO muſt be made infinite and conſequently $fA = -a$. Hence $gB = \frac{-a+AB}{-a+AB-b} \times -b$, which being made infinite, that the rays may go parallel between the glaſſes A, B, gives $-a+AB-b=o$ or $AB=a+b$. Hence whatever be the interval BC, we have $hC = -c$, and conſequently $iD = \frac{-c+CD}{-c+CD-d} \times -d$, which being made infinite, that the rays may emerge parallel, gives $-c+CD-d=o$, or $CD=c+d$. Now when the four glaſſes were concaves the ratio of the apparent magnitude to the true, or of OP to $O\Pi$, was $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+$ [a] Art. 273.
$\frac{1}{d}+\frac{BC}{bc}+\frac{BD}{bd}+\frac{CD}{cd}+\frac{BCD}{bcd}$ [a]; or $-\frac{bcd}{a}$ to $D\overline{C+d+c}$, $C\overline{B+b}+D\overline{C+d}$, c; by a like reduction to that in cor. 14; or in four convex glaſſes, as $-\frac{bcd}{a}$ to $D\overline{C-d-c}$, $C\overline{B-b}+D\overline{C-d}$, $-d$; or becauſe $CD=c+d$, as $-\frac{bcd}{a}$ to $-cc$, or as db to ca; or putting $b=a$, as d to c, whatever be the focal diſtance of the equal glaſſes A, B. And the affirmative value of $O\Pi$ ſhews that the object will appear upright[b]. [b] Art. 250.

277. *Corol.* 16. In a microſcope compoſed of two convex glaſſes A and B, if the object PL be placed at g to be found as before[c], the rays will return to the eye in parallel lines; and then, by cor. 11. the apparent diſtance $O\Pi = a \times \overline{1-\frac{BP}{b}} = \frac{a}{b} \times \overline{b-BP} = -\frac{a}{b} \times Pb$, taking $Bb=b$. And ſo the apparent magnitude is to the true, or OP to $O\Pi$, as OP to $-\frac{a}{b}Pb$, that is in a ratio compounded of b to a and of OP to Pb: and the negative value of $O\Pi$ ſhews that the object will appear inverted. Fig. 354. [c] Art. 276.

278. *Corol.* 17. Hence again when the object is remote, as in a teleſcope, the apparent magnitude is to the true as b to a; becauſe the ratio of OP to Pb becomes a ratio of equality.

279. *Corol.* 18. Since the area of an object-glaſs is the common baſe of all the pencils that flow from the ſeveral points of an object, whether near or remote; the middle ray of every pencil will paſs ſtraight through the middle point of this glaſs. This middle point may therefore be conſidered as the focus from whence the middle rays flow upon the ſubſequent glaſs or glaſſes; and conſequently if theſe rays emerge from the laſt glaſs converging to a focus, and the pupil of the eye be placed at this focus, it will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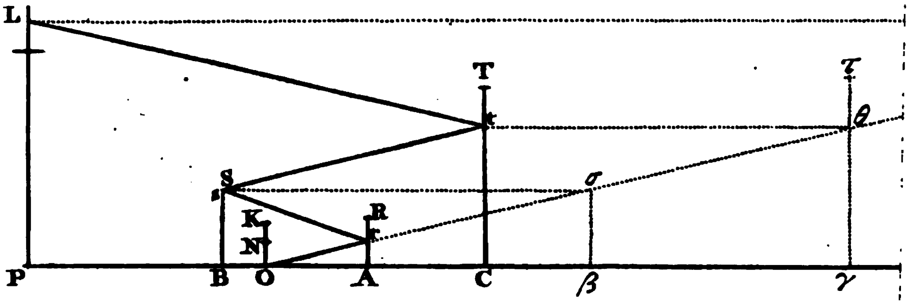
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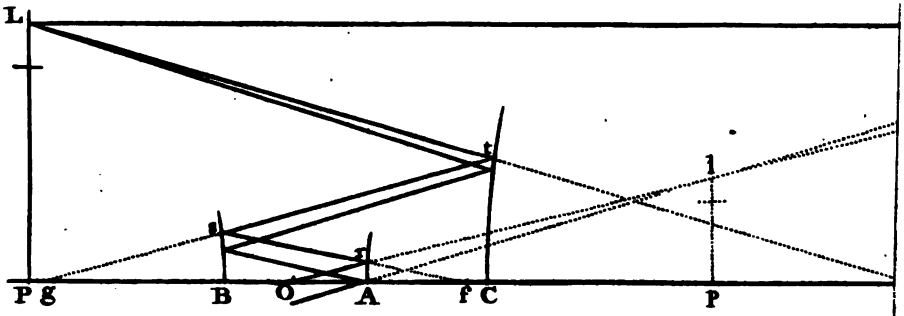
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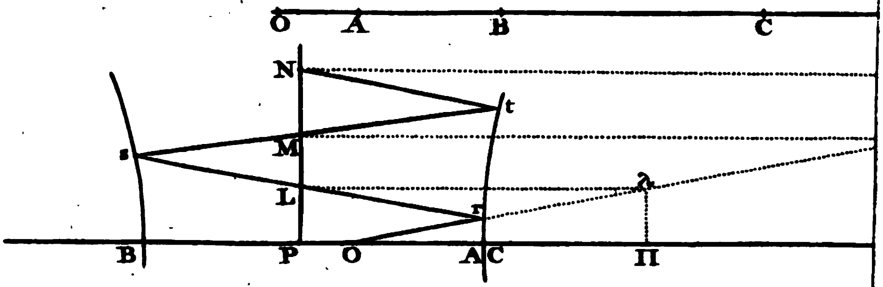
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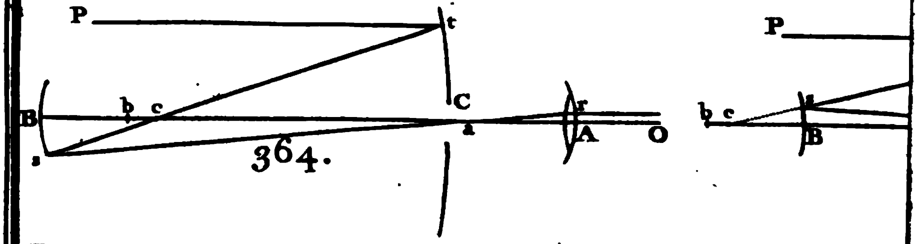
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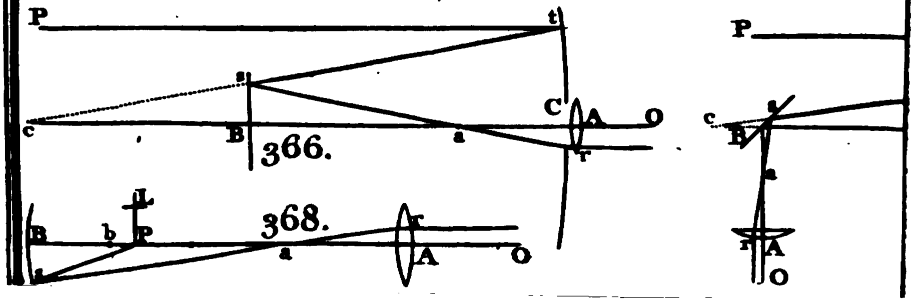
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368.

will receive all the middle rays though it were contracted to a point; and when it is more open it will alſo receive as many collateral rays of every pencil as its aperture can admit, and ſometimes the whole pencils. The viſible area of the object will therefore be the greateſt when the eye is at this focus; for in moving the pupil either way from this focus, till it comes to a place in the pencil of middle rays, where its ſection is as large as the pupil, the outmoſt of theſe rays will begin to be excluded from the pupil; and then the viſible area of the object begins to be contracted. And in like manner if the middle rays ſhall emerge diverging from a focus placed beyond the eye-glaſs, the outmoſt of them will be gradually excluded from the pupil while it recedes from that focus or from the eye-glaſs, and conſequently the viſible area will be greateſt in this caſe when the pupil is cloſe to the eye-glaſs. Now this focus O of emergent rays, when the center of the object-glaſs is the focus of the incident rays, may be found ſeveral ways. By art. 271, if B be the focus of incident rays and the glaſs A be concave, $AO = \frac{AB}{1 + \frac{AB}{a}}$; and if C be the focus of incident rays and the glaſſes B, A be concaves, $AO = \frac{AC + \frac{ABC}{b}}{1 + \frac{AB}{a} + \frac{BC}{b} + \frac{ABC}{ab}}$; and ſo on; and if any of the glaſſes be convex the ſigns of their focal diſtances muſt be changed. For example in *Galileo*'s teleſcope where the glaſs A is concave and $AB = b - a$, we have $AO = a - \frac{aa}{b}$, where a being bigger than $\frac{aa}{b}$, AO is affirmative and conſequently lyes the contrary way from A to the courſe of the rays[a], which therefore emerge from A diverging from O; and ſo the viſible area will be the greateſt when the eye is put cloſe to the glaſs A. In the aſtronomical teleſcope $AO = -a - \frac{aa}{b}$, by making the line a negative upon account of the convex eye-glaſs. Here the point O lyes on the outſide of the teleſcope a little farther from the eye-glaſs than its principal focus, the ſmall overplus $\frac{aa}{b}$ being to a as a to b. In the teleſcope conſiſting of two convex eye-glaſſes ſo placed that $AB = a + 2b$ and $BC = 2b + c$[b], and conſequently $AC = a + 4b + c$; by ſubſtituting theſe values in the rule above, we have $AO = -a - \frac{aa}{b} - \frac{aa}{c}$; or when $b = a$, $AO = -2a - \frac{aa}{c}$. Here the place of the eye is not much farther from A than twice its focal diſtance.

Fig. 355 to 358.

Fig. 355.

Fig. 356.

Fig. 357.

280. *Corol.* 19. The laſt image pl is to the object PL, as Op, the diſtance of the image from the eye, to $O\Pi$, the apparent diſtance of the object. For the triangles plO, $\Pi \Lambda O$ are ſimilar, and $\Pi \Lambda$ is equal to PL. Fig. 345.

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[a] Art. 270.

[b] Art. 275.

LEMMA.

281. Imagine a pencil of rays after ſeveral ſucceſſive reflections and refractions at ſeveral ſurfaces, to belong to ſeveral ſucceſſive focuſes, as in teleſcopes; then if any part of this pencil be ſtopt by an obſtacle of any ſhape and in any place, the other part that is not ſtopt will belong to the ſame ſucceſſive focuſes as the whole did before. Conſequently when ſeveral ſucceſſive images are formed by the ſucceſſive focuſes of ſeveral pencils, their places, ſhapes and magnitudes will continue the ſame, after any parts of thoſe pencils are ſtopt, as they were before. Therefore to determine the focuſes and images formed by ſuch partial pencils, we may juſtly argue upon any lines of a pencil along which rays might paſs, as if they really did paſs, or as if theſe lines had the properties of rays; and all the concluſions will be the ſame in both caſes, excepting thoſe concerning apparent brightneſs.

PROPOSITION II.

282. *Having the focal diſtances and apertures of any number of reflecting ſurfaces, either concave or convex, placed at any diſtances from each other and from the eye and object; it is required to find the apparent diſtance, magnitude, ſituation, degree of diſtinctneſs and brightneſs of the object ſeen by rays ſucceſſively reflected from all the ſurfaces; together with the greateſt angle of viſion and viſible area of the object, and the particular ſurface whoſe aperture limits them both.*

Ap. diſtance. Fig. 359. 283. Let the object PL be ſeen by rays, which in returning back from the eye at O to the object at P, are ſucceſſively reflected from the ſpherical ſurfaces A, B, C, whoſe focal diſtances are the lines a, b, c and common axis is the line $OABCP$. Take a line

$$O\Pi = OA + AB + BC + CP + \frac{OA,\overline{AB+BC+CP}}{a} + \frac{\overline{OA+AB},\overline{BC+CP}}{b} + \frac{\overline{OA+AB+BC},CP}{c} + \frac{OAB,\overline{BC+CP}}{ab} + \frac{OA,\overline{AB+BC},CP}{ac} + \frac{\overline{OA+AB},BCP}{bc} + \frac{OABCP}{abc};$$

and let the terms that are applyed to the focal diſtances of any odd number of concave ſurfaces be looked upon as negative, otherwiſe as affirmative; and the line $O\Pi$ will be the apparent diſtance of the object.

Ap. magnitude and ſituation. 284. And its apparent magnitude will be to the true as OP to $O\Pi$. And if the value of $O\Pi$ be affirmative the object will appear upright, otherwiſe inverted.

Ap. diſtinctneſs. 285. When $O\Pi$ is affirmative place it before the eye, otherwiſe behind it; then let the eye be removed from O to A, that its diſtance from the next ſurface may vaniſh; and here let $A\varpi$ be the apparent diſtance of the object PL to be found and placed as above directed; then let Ap be to $A\varpi$

as AO to the difference of $O\Pi$ and $A\varpi$ if they both lye the ſame way from O and A, otherwiſe to their ſum; and let the order of the points A, p, ϖ be the ſame as of the points O, p, Π; and from the ſituation of this point p a judgment may be formed of the degree of diſtinctneſs with which the object will appear. For the rays flowing from the point P, will be diſpoſed, after all the reflections, to fall upon the eye tending from the point p, when it is before the eye, or towards the point p, when it is behind the eye.

Viſual angle. Fig. 360.

286. Let the lines AR, BS, CT be the ſemidiameters of the given apertures of the ſurfaces A, B, C, and let $O\beta$ be the apparent diſtance of the line BS ſeen by reflection from the glaſs A, and $O\gamma$ the apparent diſtance of the line CT ſeen by reflections from the glaſſes B and A, to be found as above; erect the perpendiculars $\beta\sigma$ equal to BS, and $\gamma\tau$ equal to CT; and then the leaſt of the angles which any one of the perpendiculars AR, $\beta\sigma$, $\gamma\tau$ ſubtends at O, will be half the greateſt angle of viſion.

Viſible area.

287. Let this viſual angle be $\beta O\sigma$, and let $O\sigma$ produced cut a perpendicular to the axis at Π in Λ, and PL taken equal to $\Pi\Lambda$ will be the ſemidiameter of the greateſt area of the object that can be ſeen at one view by the eye at O: and therefore $\Pi\Lambda$ the ſemidiameter of this viſible area, will be to $\beta\sigma$ or BS, the ſemidiameter of the aperture which limits it, as $O\Pi$ the apparent diſtance of that area, to $O\beta$ the apparent diſtance of that aperture.

Where limited.

288. And by the ſuppoſition that $\beta O\sigma$ is the leaſt of all the angles ſubtended at O by the given lines AR, $\beta\sigma$, $\gamma\tau$, it will appear that B is the glaſs whoſe aperture confines the viſion.

Ap. brightneſs.

289. The determination of the apparent brightneſs of the point P is alſo the ſame as in the foregoing propoſition, and may be deſcribed in this other manner. If another eye at P can ſee the whole pupil of the eye at O or more by reflection from the ſame glaſſes, the point P will appear as bright to the eye at O as if the glaſſes were removed; but if the eye at P can ſee but a part of the pupil at O, the point P will appear leſs bright to the eye at O, than before, in the ſame proportion as the viſible part of the pupil is leſs than the whole; ſuppoſing no part of the incident rays to be intercepted or loſt by the reflections, or but an inſenſible one. Now the viſible area of the eye at O ſeen from the point P may be determined as above.

Fig. 361, 362.

290. The demonſtration of this propoſition is juſt the ſame as that of the foregoing propoſition, only by changing the words concave lenſes for convex ſurfaces; and refractions for reflections; and by referring to the 207th article inſtead of the 239th; and if the intervals OA, AB, BC, CP, taken according to the courſe of the rays, be joined together in one continued line $OABCP$, it appears by inſpection of the theorem for the apparent diſtance, that its expreſſion by the parts of this continued line

will be the very ſame as that of the theorem for lenſes; namely $O\Pi = OP + \frac{OAP}{a} + \frac{OBP}{b} + \frac{OCP}{c} + \frac{OABP}{ab} + \frac{OACP}{ac} + \frac{OBCP}{bc} + \frac{OABCP}{abc}$. The corollaries in the former propoſition concerning apparent brightneſs are alſo applicable to this propoſition.

291. *Corol.* 1. If any of the reflecting ſurfaces be planes, they may be conſidered as portions of ſpherical ſurfaces whoſe diameters and focal diſtances are infinite; and then the terms in the value of the apparent diſtance, which are applyed to theſe infinite focal diſtances, will vaniſh. Thus if the ſurface B be a reflecting plane, then $O\Pi = OA + AB + BC + CP + \frac{OA, \overline{AB + BC + CP}}{a} + \frac{\overline{OA + AB + BC}, CP}{c} + \frac{OA, \overline{AB + BC}, CP}{ac}$. And if the ſurfaces A, B and C be all planes, $O\Pi = OA + AB + BC + CP$; which is the ſum of all the lines deſcribed by the reciprocal motion of the neareſt ray to the axis, in paſſing between the eye and object.

Fig. 363. 292. *Corol.* 2. When rays in flowing from the eye, fall ſeveral times upon an object $PLMN$, after ſeveral reflections from two ſurfaces A, B, the object will appear at as many ſeveral diſtances. As if the ſurfaces A and B be both convex, then after one reflection at A, $O\Pi = OA + AP + \frac{OAP}{a}$; and after two reflections at A and B, $O\Pi 2 = OA + AB + BP + \frac{OA, \overline{AB + BP}}{a} + \frac{\overline{OA + AB}, BP}{b} + \frac{OABP}{ab}$; and after three reflections at A, B and A, $O\Pi 3 = OA + AB + BA + AP + \frac{OA, \overline{AB + BA + AP}}{a} + \frac{\overline{OA + AB}, \overline{BA + AP}}{b} + \frac{\overline{OA + AB + BA}, AP}{a} + \frac{OAB, \overline{BA + AP}}{ab} + \frac{OA, \overline{AB + BA}, AP}{aa} + \frac{\overline{OA + AB}, BAP}{ba} + \frac{OABAP}{aba}$; and ſo on. And by the foregoing corollary it appears what the apparent diſtances will be when one or both the reflecting ſurfaces are planes. And it is eaſy to be underſtood that one ſide of the object will be ſeen after every odd number of reflections, and its oppoſite ſide after every even number.

293. *Corol.* 3. By this and the foregoing propoſition it is alſo evident that all the appearances of an object will be determinable by the ſame general rules, when ſeen by rays which in ſome places of their paſſage are reflected from ſurfaces of any ſort, and in others are refracted through lenſes of any ſort. That is if any of the lines a, b or c be the focal diſtance of a concave or convex lens placed at A, B or C inſtead of a convex or concave ſurface, the general theorem for the apparent diſtance and every thing elſe will continue the ſame as before; excepting the ſubſequent courſe of the rays. The following corollaries are evident by the demonſtrations of the like corollaries in the firſt propoſition.

294.

294. *Corol.* 4. While the ſurfaces are fixt if the eye and object be ſuppoſed to change places, the apparent diſtance, magnitude and ſituation of the object will be the ſame as before.

295. *Corol.* 5. When the diſtances of the object from the eye and the ſurfaces are incomparably greater than their diſtances from one another, then $O\Pi = OP$ into $1 + \frac{OA}{a} + \frac{OA+AB}{b} + \frac{OA+AB+BC}{c} + \frac{OAB}{ab} + \frac{OA,\overline{AB+BC}}{ac} + \frac{\overline{OA+AB},BC}{bc} + \frac{OABC}{abc}$.

296. *Corol.* 6. When O and b are conjugate focuſes of a pencil of rays ſucceſſively reflected from any number of ſpherical ſurfaces A, B, C, the ratio of the angles AOr, Cbt, which any ray makes with the common axis of the ſurfaces before the firſt and after the laſt reflection, is the ſame as of 1 to $1 + \frac{OA}{a} + \frac{OA+AB}{b} + \frac{OA+AB+BC}{c} + \frac{OAB}{ab} + \frac{OA,\overline{AB+BC}}{ac} + \frac{\overline{OA+AB},BC}{bc} + \frac{OABC}{abc}$. Fig. 361.

297. *Corol.* 7. Having O the focus of incident rays to find b their conjugate focus after reflections from any number of ſpherical ſurfaces A, B, C; take Cb to $OA + AB + BC + \frac{OA,\overline{AB+BC}}{a} + \frac{\overline{OA+AB},BC}{b} + \frac{OABC}{ab}$, (the apparent diſtance of the laſt ſurface C,) as 1 to $1 + \frac{OA}{a} + \frac{OA+AB}{b} + \frac{OA+AB+BC}{c} + \frac{OAB}{ab} + \frac{OA,\overline{AB+BC}}{ac} + \frac{\overline{OA+AB},BC}{bc} + \frac{OABC}{abc}$ and obſerving the rule in the propoſition for the ſign of every line, place Cb from C contrary to the courſe of the rays reflected from C, if the ſecond and third terms of the proportion have like ſigns, otherwiſe according to that courſe, and b will be the conjugate focus to O.

298. *Corol.* 8. Hence if the reflecting ſurfaces A, B, C be all planes, $Cb = OA + AB + BC$, and lyes contrary to the courſe of the rays reflected from C.

299. *Corol.* 9. When the object and the ſurfaces A, B, C are placed at ſuch diſtances from one another, that the rays in any pencil ſhall fall upon the eye in parallel lines, then the apparent diſtance $O\Pi = -a$ into $1 + \frac{BC+CP}{b} + \frac{CP}{c} + \frac{BCP}{bc}$ or $= AB + BC + CP + \frac{AB,\overline{BC+CP}}{b} + \frac{\overline{AB+BC},CP}{c} + \frac{ABCP}{bc}$. And this apparent diſtance and conſequently the apparent magnitude, ſituation, diſtinctneſs and brightneſs of the object will be invariable wherever the eye is placed.

300. *Corol.* 10. When the rays of a pencil fall upon the eye in parallel lines, the object and ſurfaces *A*, *B*, *C* are placed at ſuch intervals that $1+\frac{AB+BC+CP}{a}+\frac{BC+CP}{b}+\frac{CP}{c}+\frac{AB,\overline{BC+CP}}{ab}+\frac{\overline{AB+BC},CP}{ac}+\frac{BCP}{bc}+\frac{ABCP}{abc}=0$; and conſequently when the object is remote, the ſurfaces are placed at ſuch intervals that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{AB}{ab}+\frac{AB+BC}{ac}+\frac{BC}{bc}+\frac{ABC}{abc}=0$; and on the contrary.

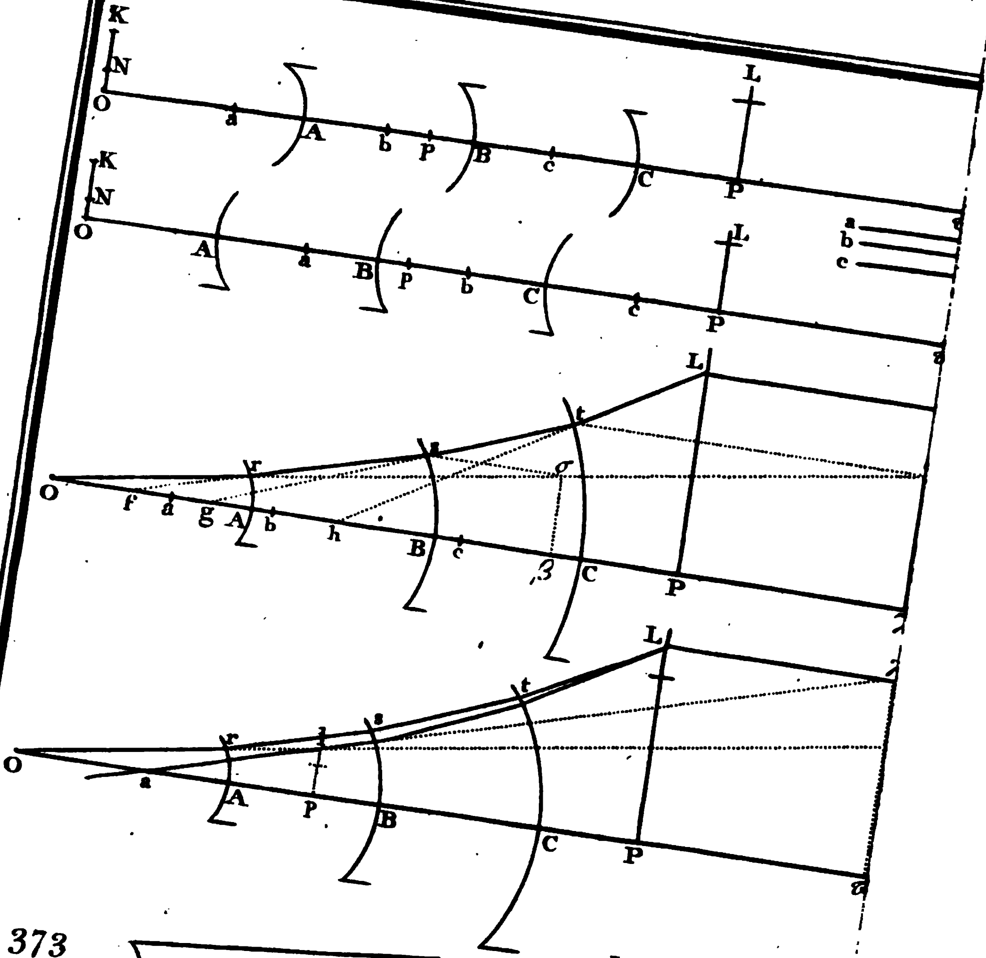
301. *Corol.* 11. In a teleſcope ſuppoſed to conſiſt of two convex ſurfaces *A* and *B*, the apparent magnitude of the object, is to the true, or OP is to $O\Pi$ as $-\frac{1}{a}$ to $\frac{1}{b}$; and in a teleſcope conſiſting of three ſuch ſurfaces *A*, *B*, *C*, OP is to $O\Pi$ as $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+\frac{BC}{bc}$; and in four ſuch ſurfaces *A*, *B*, *C*, *D*, $O\Pi$ is to OP as $-\frac{1}{a}$ to $\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{BC}{bc}+\frac{BC+CD}{bd}+\frac{CD}{cd}+\frac{BCD}{bcd}$; and ſo on.

Fig. 364, 365. 302. *Corol.* 12. In the figures for reflecting teleſcopes, let the points *a*, *b*, *c* be the principal focuſes of the reſpective given ſurfaces *A*, *B*, *C*; and when the rays in any one pencil are parallel before the firſt and after the laſt reflections at *A* and *C*, the points *a*, *c* muſt be conjugate focuſes with reſpect to the intermediate reflection of the ſame rays at the ſurface *B*. Therefore if the interval AB and conſequently the interval ab be aſſumed as moſt convenient, ſay as ba to bB ſo bB to bc and the point *c* will be determined[a] and conſequently the point *C*, and alſo the interval BC. Now if all the ſurfaces be concaves, the apparent magnitude is to the true, as $\frac{1}{a}$ to $-\frac{1}{b}-\frac{1}{c}+\frac{BC}{bc}$*, or as $\frac{bc}{a}$ to $BC-b-c$ or bc the interval of the focuſes *b* and *c*; that is in a ratio compounded of *c* to *a* and of *b* to the interval bc; which being affirmative ſhews that the object will appear upright[b]. But if the ſurface *B* be convex and only *A* and *C* be concaves, the apparent magnitude will be to the true as $\frac{1}{a}$ to $\frac{1}{b}-\frac{1}{c}-\frac{BC}{bc}$, or as $\frac{b,c}{a}$ to $c-b-BC$ or $-bc$, the interval of the focuſes *b* and *c*; that is in a ratio compounded of *c* to *a* and of *b* to bc, the interval of *b* and *c*; which being negative ſhews that the object will appear inverted. And if the interval bc, or the ratio of $\frac{b,c}{a}$ to the interval bc, be aſſumed at firſt, then the places of *A* and *a* will be determined by making bc, bB, ba continual proportionals.

[a] Art. 207.

* Art. 301.

[b] Art. 285.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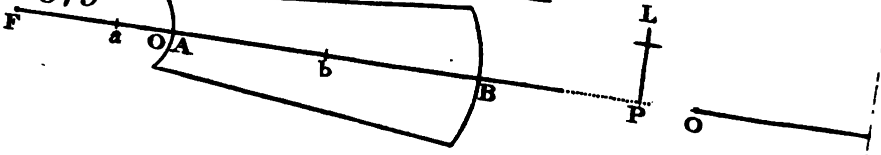
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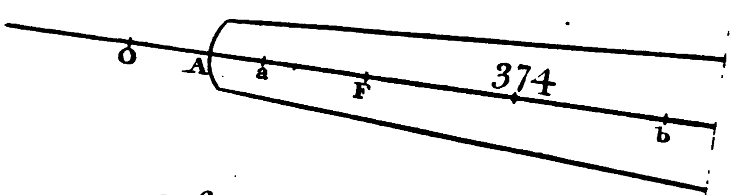
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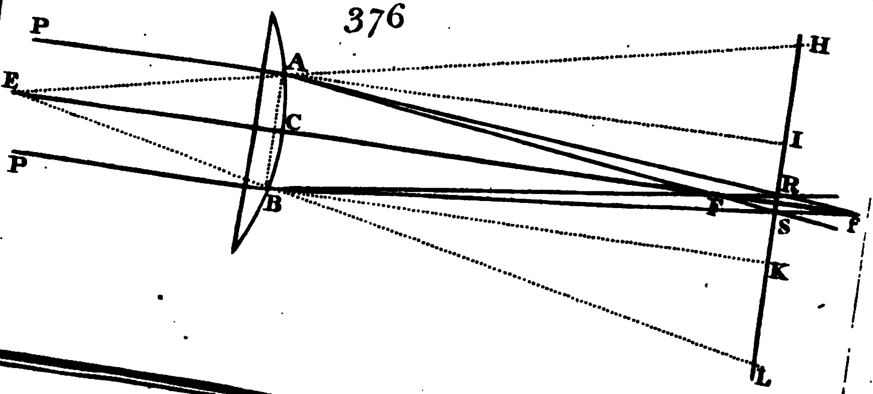
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So far in theory, which is reduced to practice by making the focal diſtance and the breadth of the ſurface *B* very ſmall, that it may not intercept too many rays, in their paſſage from the object to the large concave *C*; and by making a moderate hole in the middle of this concave at *C* for the rays to paſs through after reflection from *B*; and by ſubſtituting a convex lens in the place of the concave ſurface at *A*, ſo that the ſpectator may look towards the object. For ſuppoſing this lens to have the ſame focal diſtance as the concave ſurface at *A*, the apparent diſtance, magnitude, ſituation, diſtinctneſs and brightneſs of the object will continue the ſame as before[a].

303. *Corol.* 13. Hence if the ſpherical ſurface at *B* be changed into a plane by increaſing its ſemidiameter and focal diſtance bB to infinity, the ratio of bB to the interval bc will become a ratio of equality; and then the apparent magnitude will be to the true as c to a by the foregoing corollary. Now when this ratio of Cc to Aa is very great, the common focus a can come but a very little way within the concave Ct, though the eye-glaſs were placed in the hole at C; therefore Ba and Bc being now equal, or Bc being half of ca muſt be near half of cC; and conſequently the breadth of the plane Bs muſt be near half the breadth of the concave Ct, to receive the whole pencil reflected from Ct; and then it would alſo intercept near half the incident rays that come from the object. But if the plane Bs be turned obliquely, to reflect the rays ſideways to the eye-glaſs *A*, its diſtance from c and conſequently its breadth may be diminiſhed at pleaſure without altering the apparent magnitude; which the plane neither increaſes nor diminiſhes. For putting b infinite in cor. 9. CP or OP is to $O\Pi$ as c to $-a$. Fig. 366. Fig. 367.

304. *Corol.* 14. When the rays that flow from a near object PL, go in parallel lines to the eye after two reflections at the concave ſurfaces *B* and *A*, or after one reflection at *B* and a refraction through a convex lens at *A* whoſe focal diſtance is $-a$, the apparent diſtance of the object from the eye at any point O is $a \times \overline{1 - \frac{BP}{b}}$[*] or $\frac{a}{b} \times \overline{b - BP}$ or $-\frac{a}{b}\,Pb$; which being negative ſhews that the object appears inverted[b]. Hence the apparent magnitude is to the true, or the true diſtance to the apparent, as OP to $\frac{a}{b}\,Pb$, or in a ratio compounded of b to a and of OP to Pb. In theſe reflecting microſcopes the object PL being very ſmall can intercept but very few rays in their paſſage from *B* to *A*. Fig. 368.

305. *Corol.* 15. Hence again when the object is remote as in a reflecting teleſcope, conſiſting of one large concave and a convex eye-glaſs, the apparent magnitude of the object is to the true as b to a, becauſe the ratio of OP to Pb becomes a ratio of equality, and becauſe the reflecting plane does not alter the apparent magnitude.

[a] Art. 293.

[*] Art. 299.

[b] Art. 284.

306. *Corol.* 16. The place of the eye in reflecting teleſcopes, where the middle rays of each pencil interſect one another, may be found by making Ba, BA, BO continual proportionals[a], becauſe the reflecting ſurface B correſponds with what I have ſaid of the object-glaſs of a refracting teleſcope, in the 279th article as appears by inſpection of the 184th figure.

[a] Art. 239.

PROPOSITION III.

307. *Having the focal diſtances and apertures of any number of ſpherical ſurfaces, which intercede any given mediums, and are placed at any diſtances from each other and from the eye and object; it is required to find the apparent diſtance, magnitude, ſituation, degree of diſtinctneſs and brightneſs of an object ſeen through all the mediums: together with the greateſt angle of viſion and viſible area of the object, and the particular aperture which limits them both.*

Ap. diſtance. Fig. 369, 370.

308. Let PL be an object viewed by the eye at O through any number of ſpherical ſurfaces placed at A, B, C, whoſe centers a, b, c are all in the line OP, and whoſe focal diſtances of rays that fall parallel on their ſides next the eye are the lines a, b, c. At firſt let us ſuppoſe the ſemidiameters Aa, Bb, Cc to lye all the ſame way from their ſurfaces; and to be all ſeparate from one another and from the eye and object; and that the medium adjoining to the concave ſide of every ſurface, is rarer or leſs refractive than the medium adjoining to its convex ſide; then take a line

$$O\Pi = OP + \frac{Oa, AP}{a} + \frac{Ob, BP}{b} + \frac{Oc, CP}{c} + \frac{Oa, Ab, BP}{ab} + \frac{Oa, Ac, CP}{ac} + \frac{Ob, Bc, CP}{bc} + \frac{Oa, Ab, Bc, CP}{abc},$$

and it will be the apparent diſtance of the object.

In all other caſes the lines OP, AP, BP, CP muſt ſtill remain affirmative in this value of $O\Pi$, but any one of the lines Oa, Ob, Oc will be negative if it lyes behind the eye: ſecondly any one of the lines Ab, Ac, Bc, will be negative if it tends towards the eye from the ſurface that terminates it: thirdly any one of the focal diſtances will be negative, if the order of the denſities of the mediums remaining as in the firſt caſe, the ſemidiameter of the ſurface to which it belongs be ſituated on the other ſide of its ſurface; or if, the poſition of the ſemidiameter remaining as at firſt, the denſities of the adjoining mediums be tranſpoſed to contrary ſides of that ſurface. The ſign of every line involved in the foregoing value of $O\Pi$ being thus determined, the ſign of every term of it which involves any odd number of negative lines muſt be looked upon as negative, otherwiſe as affirmative; then will $O\Pi$, or the ſum of all the terms according to their ſigns, be the apparent diſtance of the object.

309. And the apparent magnitude of the object will be to its true magnitude as OP to $O\Pi$.

Ap. magnitude.

310. And if the value of $O\Pi$ be affirmative the object will appear upright, otherwiſe inverted.

Ap. ſituation.

311. When $O\Pi$ is affirmative place it before the eye, otherwiſe behind it. Then imagine the eye to be removed from O to a, that its diſtance from the center of the next ſurface may vaniſh; and here let $a\varpi$ be the apparent diſtance of the object, to be found and placed by the ſame rules that $O\Pi$ was; then let ap be to $a\varpi$ as aO to the difference of $O\Pi$ and $a\varpi$, if they lye the ſame way from O and A, otherwiſe to their ſum; and let the order of the points a, p, ϖ be the ſame as the order of the points O, p, Π; and from the ſituation of this point p a judgment may be formed of the degree of diſtinctneſs with which the object will appear. Becauſe p is the place of the laſt image of the object.

Ap. diſtinctneſs.

312, The determinations of the viſual angle and viſible area and of the aperture which limits them, are the ſame as in the propoſition for lenſes [a].

Viſual angle and viſ. area.

313. Since the magnitude of the pupil is ſubject to be varied by various degrees of light, let NO be its ſemidiameter, when the object PL is viewed by the naked eye from the diſtance OP; and upon a plane that touches the eye at O, let OK be the ſemidiameter of the greateſt area viſible to another eye at P through all the apertures, to be found as PL was; or which is the ſame thing, let OK be the ſemidiameter of the greateſt area inlightened by a pencil of rays flowing from P through all the apertures; and when this area is not leſs than the area of the pupil, the apparent brightneſs of the point P ſeen by the refracted rays, will be to its apparent brightneſs ſeen through an uniform medium by unrefracted rays, in a ratio compounded of the apparent magnitude, to the true magnitude of any ſurface at O ſeen from P, and of the true magnitude, to the apparent, of any ſurface at P ſeen from O: but if the inlightened area at O be leſs than the area of the pupil, the apparent brightneſs of P ſeen by the refracted rays, will be to its apparent brightneſs ſeen by the unrefracted rays, in a ratio compounded of the two former ratios, and of this inlightened area, to the area of the pupil. This would be the proportion of the apparent brightneſs if all the rays were tranſmitted through the mediums, or if only an inſenſible part of them was ſtopt by reflections at the ſurfaces and by the opacity of the matter.

Ap brightneſs.

[a] Art. 252.

DEMONSTRATION.

314. The firſt part of the demonſtration of the firſt of theſe propoſitions[b] gives $O\Pi = OA \times \frac{fA+AB}{fA} \times \frac{gB+BC}{gB} \times \frac{hC+CP}{hC}$. By which theorem $O\Pi$ will be given ſo ſoon as fA, gB, hC can be found. Theſe may be

Ap. diſtance.
Fig. 371.

[b] Art. 256.

found by the 238th article. For ſuppoſing any line AB to be the focal diſtance of the ſurface A when the rays fall parallel on its ſide next the object, we have OB to Oa as OA to Of, and disjointly OB to aB as OA to fA; therefore ſince we ſuppoſed the line a to be the other focal diſtance of this ſurface and conſequently to be equal to aB[a], we have $fA = \frac{OA, a}{Oa + a}$; $gB = \frac{\overline{fA + AB} \times b}{fA + Ab + b}$; $hC = \frac{\overline{gB + BC} \times c}{gB + Bc + c}$. Whence it is eaſy to collect, that if the eye at O views an object at B through one ſurface at A, its apparent diſtance $O\beta = OB + \frac{Oa, AB}{a}$; that if the eye at O views an object at C through two ſurfaces at A, B, its apparent diſtance $O\gamma = OC + \frac{Oa, AC}{a} + \frac{Ob, BC}{b} + \frac{Oa, Ab, BC}{ab}$; that if the eye at O views an object PL through three ſurfaces at A, B, C, its apparent diſtance $O\Pi = OP + \frac{Oa, AP}{a} + \frac{Ob, BP}{b} + \frac{Oc, CP}{c} + \frac{Oa, Ab, BP}{ab} + \frac{Oa, Ac, CP}{ac} + \frac{Ob, Bc, CP}{bc} + \frac{Oa, Ab, Bc, CP}{abc}$. The rules given for determining the ſigns of every line in this theorem in all other caſes, will be evident by obſerving that the points O, A, B, C, P are fixt in order and poſition in all caſes; that the figures of the ſurfaces and the poſition of their centers may be changed from thoſe in the firſt caſe to ſuch as are propoſed in any other, by increaſing their ſemidiameters Aa, Bb, Cc till they become infinite, and then negative if need be; that any of their focal diſtances will become infinite and then negative, either when the ſemidiameter of its ſurface becomes infinite and then negative, or elſe when the denſity of one of the contiguous mediums is gradually altered till it equals the other, and ſtill more till it differs from the other the contrary way; and laſtly that during theſe gradual alterations any line will become negative after it has been nothing or infinite any odd number of times, in paſſing from its ſtate in the firſt caſe to the ſtate propoſed in any other.

[a] Art. 224.

The determination of the apparent magnitude is evident by art. 141; and that of the apparent ſituation by the latter part of art. 139.

Ap. diſtinctneſs.
Fig. 372.

315. Compleat the rectangle $LP\varpi\lambda$ and join $a\lambda$ meeting $O\Lambda$ in l, and the line lp drawn perpendicular to the axis of the ſurfaces will be the laſt image of the object LP. Becauſe the ſame point L is ſeen by a ray which falls upon the eye at O in the direction ΛlO, and alſo by a ray which falls upon the eye at a in the direction λla; and therefore the point l, where the lines $O\Lambda$, $a\lambda$ croſs one another, is the focus of the emergent rays. Now ſince the triangles apl, $a\varpi\lambda$, and alſo Opl, $O\Pi\Lambda$ are equiangular, we have ap to $a\varpi$ as (pl to $\varpi\lambda$ or $\Pi\Lambda$, or as) Op to $O\Pi$, or as

$Op \mp ap$ or Oa to $O\Pi \mp a\varpi$, according as p falls without or within the line Oa, and conſequently according as $O\Pi$ and $a\varpi$ lye the ſame way or contrary ways from O and a. And the order of the points a, p, ϖ is the ſame as the order of the points a, l, λ, or of the points O, l, Λ or of the points O, p, Π.

Ap. brightneſs. Fig. 347, 348.

316. Firſt if OK be not leſs than ON, the area of the pupil will be totally inlightened by the pencil that flows from P. Let $PtsrN$ be a ray of that pencil cutting the ſurface Ct in t; and ſuppoſing the refracting ſurfaces were all removed, let an unrefracted ray PMN cut the line Ct in M. Then the quantity of refracted light which falls upon the line NO, is to the quantity of unrefracted light which would fall upon it, as the angle CPt to the angle CPM, that is as the apparent magnitude of the line NO, to its true magnitude ſeen by an eye from P. And therefore by turning the figure round about the axis OP, the quantity of refracted light which fills the pupil, is to the quantity of unrefracted light which would fill it, as the apparent magnitude of any ſurface at O, to the true. Therefore ſince the real brightneſs of any portion or phyſical point of the retina, is directly as the quantity of light which falls upon it, and inverſely as its own magnitude; the apparent brightneſs of the point P ſeen by the refracted rays, is to its apparent brightneſs ſeen by the unrefracted rays in a ratio compounded of the apparent magnitude, to the true magnitude, of any ſurface at O ſeen from P, and of the true magnitude, to the apparent, of any ſurface at P ſeen from O. Now if OK be leſs than ON, then by ſuppoſing a leſſer pupil whoſe ſemidiameter is OK, we have ſhewn that the apparent brightneſs of P ſeen by refracted rays paſſing through this ſmaller pupil, (which is the ſame as if they paſſed through the larger,) is to its apparent brightneſs ſeen by unrefracted rays paſſing through it, in the given ratio compounded of thoſe abovementioned; and this latter apparent brightneſs of P ſeen by unrefracted rays paſſing through the ſmaller pupil, is to its apparent brightneſs ſeen alſo by unrefracted rays paſſing through the larger, in the ratio of the ſmaller pupil to the larger; which ratio being compounded with the former gives the ratio required.

Fig. 369, 370.

317. *Corol.* 1. When the object is ſo remote that the diſtances OP, AP, BP, CP may be conſidered as equal to one another, then the apparent diſtance $O\Pi = OP$ into $1 + \frac{Oa}{a} + \frac{Ob}{b} + \frac{Oc}{c} + \frac{Oa, Ab}{ab} + \frac{Oa, Ac}{ac} + \frac{Ob, Bc}{bc} + \frac{Oa, Ab, Bc}{abc}$.

Fig. 371.

318. *Corol.* 2. Hence when O and b are conjugate focuſes of a pencil of rays refracted through any number of ſurfaces A, B, C, the ratio of

the angles AOr, Cbt, made by the incident and emergent parts of any ray with the axis of the ſurfaces, is the ſame as of 1 to $1 + \frac{Oa}{a} + \frac{Ob}{b} + \frac{Oc}{c} + \frac{Oa, Ab}{ab} + \frac{Oa, Ac}{ac} + \frac{Ob, Bc}{bc} + \frac{Oa, Ab, Bc}{abc}$. For this latter ratio is the ſame as the ratio of OP to $O\Pi$, by the firſt corol., that is of the apparent magnitude of a remote object ſeen from O, to its true magnitude ſeen from O or b by the naked eye, or as the angle at O to the angle at b.

319. *Corol.* 3. Hence if O be the focus of incident rays, the focus b of the emergent rays from the laſt ſurface C, may be found by taking Cb to $O\gamma$ or $OC + \frac{Oa, AC}{a} + \frac{Ob, BC}{b} + \frac{Oa, Ab, BC}{ab}$ as 1 to $1 + \frac{Oa}{a} + \frac{Ob}{b} + \frac{Oc}{c} + \frac{Oa, Ab}{ab} + \frac{Oa, Ac}{ac} + \frac{Oa, Bc}{bc} + \frac{Oa, Ab, Bc}{abc}$; and by placing Cb contrary to the courſe of the rays if the ſecond and third terms of this proportion have like ſigns, otherwiſe according to their courſe. For compleating the rectangle $\gamma Ct\theta$, Cb is to $O\gamma$ as the angle $\gamma O\theta$ to the angle Cbt[a], that is in the ratio abovementioned, by cor. 2.

320. *Corol.* 4. When the diſtances of the ſurfaces and object from one another, and their focal diſtances a, b, c are ſuch, that $1 + \frac{AP}{a} + \frac{BP}{b} + \frac{CP}{c} + \frac{Ab, BP}{ab} + \frac{Ac, CP}{ac} + \frac{Bc, CP}{bc} + \frac{Ab, Bc, CP}{abc} = 0$, the rays of any pencil will fall upon the eye in parallel lines; and then the apparent diſtance $O\Pi$ will be equal to $a\varpi = aP + \frac{ab, BP}{b} + \frac{ac, CP}{c} + \frac{ab, Bc, CP}{bc}$ and this apparent diſtance being invariable, the apparent magnitude, ſituation and degree of diſtinctneſs and brightneſs will alſo be invariable wherever the eye is placed. For the rays will fall parallel upon the eye when the lines $O\Lambda$, $a\lambda$ are parallel, and conſequently when $O\Pi = a\varpi$ or $O\Pi - a\varpi = 0$; and the value of $a\varpi$ is found by making Oa vaniſh in the general value of $O\Pi$[b].

321. *Corol.* 5. Hence when the object is ſo remote that the diſtances AP, BP, CP, may be conſidered as equal to one another, the rays which come parallel upon the firſt ſurface will emerge parallel from the laſt, if the ſurfaces be ſo diſpoſed that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{Ab}{ab} + \frac{Ac}{ac} + \frac{Bc}{bc} + \frac{Ab, Bc}{abc} = 0$; and on the contrary: and then $O\Pi$ (or $a\varpi$) $= OP$ into $1 + \frac{ab}{b} + \frac{ac}{c} + \frac{ab, Bc}{bc}$. For example let AB be the axis of a ſolid medium, as of glaſs, and F be the common focus of rays that come parallel upon one of its ſurfaces and go parallel from the other; and when the ſurface A is concave and B convex, by changing the ſign of b (becauſe the order of the denſities of the

Fig. 373.

[a] Art. 222.

[b] Art. 308.

the adjoining mediums is changed) the former of theſe equations gives $\frac{1}{a} - \frac{1}{b} - \frac{Ab}{ab} = 0$; whence $b = a + Ab = FA + Ab = Fb$; and the latter equation gives $OP : O\Pi :: b : b - ab :: Fb : Fb - ab$ or $Fa :: bB : aA$* $:: FB : FA$. And when the ſurface A is convex the ſame proportion comes out by changing the ſign of a. Thus teleſcopes might be made of one continued ſolid; but to magnify much they muſt be made pretty long, and conſequently but few rays could be tranſmitted through ſo great a length of glaſs; as is found by experience. * Art. 224. Fig. 374.

322. *Corol.* 6. Let aF be the focal diſtance of a ſphere whoſe center is a or b and diameter AB, and the object PL will appear through it at the diſtance $O\Pi = OP - \frac{OaP}{aF}$ if the ſphere be denſer than the ambient medium, or at the diſtance $O\Pi = OP + \frac{OaP}{aF}$ if it be rarer: and theſe expreſſions are the ſame as in a lens of no thickneſs[a]. For in the general value of $O\Pi = OP + \frac{Oa, AP}{a} + \frac{Ob, BP}{b} + \frac{Oa, Ab, BP}{ab}$; inſtead of a, b, Ob, Ab, put $-a, -b, Oa, Aa$, reſpectively[b]; and in the denſer ſphere we have $O\Pi = OP - \frac{Oa, AP}{a} - \frac{Oa, BP}{b} + \frac{Oa, Aa, BP}{ab} = OP - Oa \times \overline{\frac{AP}{a} + \frac{BP}{b} - \frac{Aa, BP}{ab}} = OP - Oa \times \overline{\frac{AP}{a} + \frac{BP}{a} \times \frac{a - Aa}{b}} = OP - Oa \times \frac{AP + BP}{a} = OP - Oa \times \frac{2aP}{a} = OP - \frac{OaP}{aF}$ because $aF = \frac{1}{2} a$, as is eaſy to collect from the 227th article or from cor. 3. And in the rarer ſphere the ſign of aF muſt be changed. Fig. 375. a Art. 248. b Art. 308.

323. *Corol.* 7. Let an object PL be viewed by the eye at O through any number of given mediums diſtinguiſhed from one another by parallel planes Ar, Bs, Ct; and when a ray goes from the eye to the object let the ſine of incidence be to the ſine of refraction at the plane Ar as q to r, at the plane Bs as r to s, at the plane Ct as s to t; then take a line $O\Pi = OA + \frac{r}{q} AB + \frac{s}{q} BC + \frac{t}{q} CP$; and the object will appear at the ſame diſtance, of the ſame magnitude, in the ſame ſituation and with the ſame degree of diſtinctneſs and brightneſs through all the mediums as it would do when viewed in one uniform medium from the diſtance $O\Pi$. Fig. 344.

This may be gathered from the general theorem for $O\Pi$, when varied a little by introducing the ſemidiameters of the ſurfaces inſtead of their focal diſtances, whoſe relation is given by the 224th article, and then by making the ſemidiameters infinite; but more readily as follows. By art. 223. we have $fA : OA :: q : r$; $gB : fA + AB :: r : s$; $hC : gB + BC :: s : t$. Hence $fA = \frac{q}{r} OA$; $gB = \frac{r}{s} \times \overline{fA + AB} = \frac{q}{s} OA + \frac{r}{s} AB$;

AB; $bC = \frac{s}{t} \times \overline{gB + BC} = \frac{q}{t} OA + \frac{r}{t} AB + \frac{s}{t} BC$. But $O\Pi = OA \times \frac{fA + AB}{fA} \times \frac{gB + BC}{gB} \times \frac{bC + CP}{bC} = OA \times \overline{1 + \frac{AB}{fA}} \times \overline{1 + \frac{BC}{gB}} \times \overline{1 + \frac{CP}{bC}}$ as before[a]. Whence it is eaſy to collect, that if the eye at O views an object at B through one plane Ar, its apparent diſtance $O\beta = OA + \frac{r}{q} AB$; that if the eye at O views an object at C through two planes Ar, Bs, its apparent diſtance $O\gamma = OA + \frac{r}{q} AB + \frac{s}{q} BC$; that if the eye at O views an object at P through three planes Ar, Bs, Ct, its apparent diſtance $O\Pi = OA + \frac{r}{q} AB + \frac{s}{q} BC + \frac{t}{q} CP$; and ſo on. To find the place of the laſt image, put $OA = 0$, then $A\varpi = \frac{r}{q} AB + \frac{s}{q} BC + \frac{t}{q} CP$; by which it appears that ΠA and $\varpi\lambda$, and conſequently the laſt image pl do all coincide in place and magnitude; which makes this corollary very evident, ſuppoſing as to the brightneſs that no rays are reflected from the ſurfaces; and that all the mediums are equally tranſparent.

a. Art. 314.

Fig. 345.

CHAPTER VI.

To determine the aberrations of rays, from the geometrical focus, cauſed by their unequal refrangibility, and alſo by the ſphericalneſs of the figure of reflecting and refracting ſurfaces.

PROPOSITION I.

324. LET *the common ſine of incidence be to the ſine of refraction of the leaſt refrangible rays as* I *to* R, *and to the ſine of refraction of the moſt refrangible rays as* I *to* S; *and the diameter of the leaſt circular ſpace into which heterogeneal parallel rays can be collected, by a ſpherical ſurface or by a plano-convex lens, will be to the diameter of its aperture, in the conſtant ratio of* S − R *to* S + R − 2 I.

Fig. 376. For let an heterogeneal ray PA fall upon a ſpherical ſurface ACB, and let it be ſeparated by refraction into the rays AF, Af, cutting the axis EC, drawn parallel to PA, in F and f. Take the arch CB equal to CA and let another heterogeneal ray PB, coming parallel to PA be refracted into the lines BF, Bf, cutting the two former rays in R and S. Join RS and produce it till it meets the incident rays produced in I and K, and the perpendiculars EA, EB to the refracting ſurface at the points A, B, in

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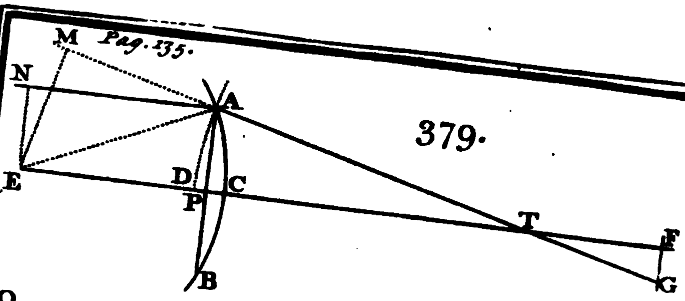
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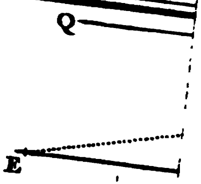

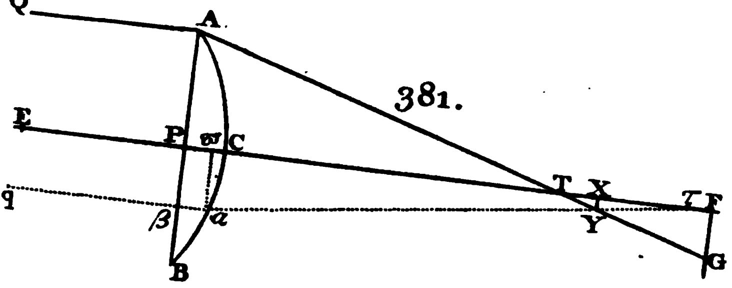
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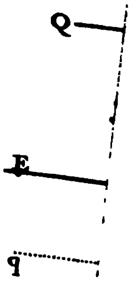

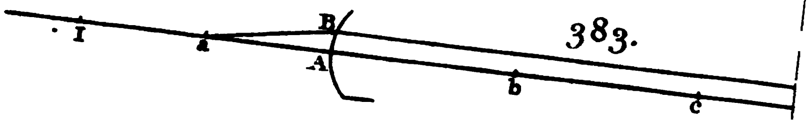
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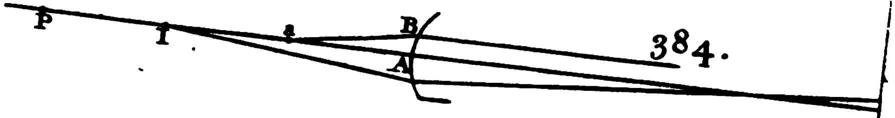
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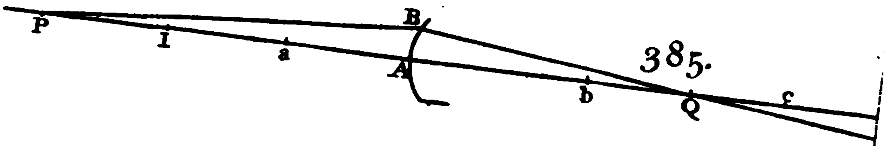
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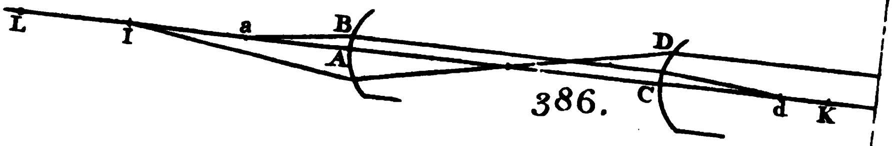
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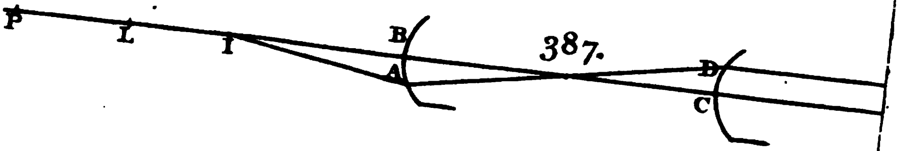
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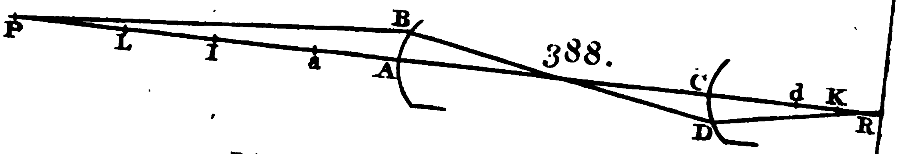
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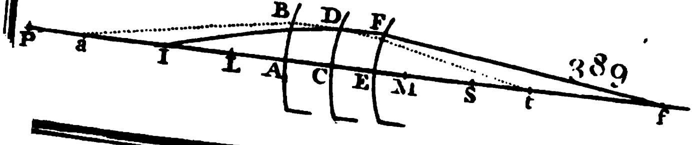
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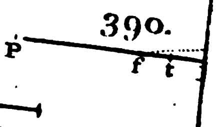
390.

in H and L. And when AB, the breadth of the aperture or of the pencil, is but moderate, and conſequently the refractions at A, B but ſmall, the angles of incidence and refraction, HAI, HAR, HAS, or the arches that meaſure them, or their perpendicular ſubtenſes HI, HR, HS, will be to each other very nearly in the ſame given ratios as thoſe of the ſines I, R, S of thoſe angles[a]. And disjointly the differences of thoſe ſubtenſes will be proportionable to the differences of theſe ſines: that is, the line $RS : RI :: S-R : R-I$, and doubling the conſequents, $RS : 2RI$ or $IK - RS :: S-R : 2R-2I$; and conjointly $RS : IK$ or $AB :: S-R : S+R-2I$. From this given ratio of RS to AB in which they increaſe or decreaſe together, it appears that all the intermediate rays which fall upon AB will paſs through RS. And when parallel rays fall perpendicularly upon the plane ſide of a plano-convex lens, they are refracted only at their emergence from its convex ſurface; and ſo the aberrations are the ſame in both caſes. *Q. E. D.*

325. *Corol.* 1. Hence the diameter RS, of the circle of aberrations that contains all the incident rays, is a 55th part of the diameter AB of the aperture of a plano-convex glaſs, whatever be its focal diſtance. For ſuppoſing AR and AS to be the outmoſt red and indigo rays, their ſines of incidence and refractions I, R, S are to each other as 50, 77, 78[b]. Whence $S-R$ is to $S+R-2I$ as 1 to 55.

326. *Corol.* 2. The diameter of the leaſt circle that can receive the rays of any ſingle colour or of ſeveral contiguous colours is alſo determinable from the proportions of their ſines. Thus all the orange and yellow is contained in a circle whoſe breadth is the 260th part of the breadth of the aperture of the plano-convex glaſs; the ſines of the outermoſt orange AR and yellow AS being to the common ſine of incidence as $77\frac{1}{8}$ and $77\frac{1}{3}$ to 50[c].

327. *Corol.* 3. In different ſurfaces, or plano-convex glaſſes, the angles of aberration RAS are as the breadths of the apertures AB directly and as the focal diſtances CF inverſely; becauſe any angle, as RAS, is as its ſubtenſe RS directly and as its radius AR or CF inverſely.

LEMMA.

Fig. 377, 378.

328. The verſed ſines AB, AC of very ſmall arches BD, CD, of unequal circles BDG, CDH, that have the ſame right ſine AD, are reciprocally proportionable to their diameters BG, CH very nearly; that is $AB : AC :: CH : BG$.

For ſince the rectangles under BAG and CAH are each equal to the ſquare of AD[d], and conſequently to each other; their ſides are reciprocally proportionable[e], that is AB is to AC as AH to AG or as CH to BG very nearly, when the verſed ſines are incomparably leſs than the diameters themſelves[f], *Q. E. D.*

PROPO-

[a] Art. 204. 220.

[b] Art. 179.

[c] Art. 179.

[d] Euc. VI. 13.

[e] Euc. VI. 14.

[f] Art. 204.

PROPOSITION II.

Fig. 379.

329. *When homogeneal parallel rays* NA, EC *fall upon a ſpherical ſurface* AC *whoſe center is* E, *the longitudinal aberration* FT *of any refracted ray* AT *from* F *the focus of the pencil, is to the verſed ſine of the arch* AC *intercepted between the point of incidence and the axis* ECF, *in the given ratio of the ſquare of the ſine of refraction, to the rectangle under the ſine of incidence and the difference of the ſines very nearly; and the aberration is the ſame when the rays fall perpendicularly upon the plane ſide of a plano-convex lens.*

For when the refraction is made in the paſſage of a ray *NA* from a denſer to a rarer medium, it appears by the deſcription of a cauſtick in the 74th article, that the interſection *T* of the refracted ray *AT* with the axis *ECF*, lyes between the refracting ſurface and its focus *F*. With the center *T* and ſemidiameter *TA* having deſcribed the arch *AD* cutting the axis in *D*, draw the ſine *AP* of the arches *AC*, *AD*, and alſo *EN* and *EM* the ſines of incidence and refraction, for which put *n* and *m*; then becauſe the triangles *ETM*, *ATP* are ſimilar, it will be as $ET : TA$ or
* Art. 225. $TD :: (EM : AP$ or $EN ::) EF : FC$*; and disjointly $TF : EF :: (FC -$
TD or$) TF - CD : FC$; and alternately $TF : TF - CD :: EF : FC$;
a Art. 224. and disjointly $TF : CD :: (EF : EC ::) m : m - n$[a]. Again ſince $(PD :$
b Art. 328. $PC :: CE : DT$[b] or FC[c], and conjointly) $CD : CP :: (EF : FC ::) m :$
c Art. 204. n; by compounding this and the foregoing proportion, it will be as TF
$: CP :: mm : \overline{m-n}, n$. *Q. E. D.*

330. *Corol.* 1. The ſegment *ACBPA* may be conſidered as a plano-convex lens; and when rays fall parallel upon its plane ſide, the longitudinal aberration of the extream ray falling upon *A* is equal to $\frac{9}{2}$ of its thickneſs *PC*, as appears by putting 3 and 2 for *m* and *n* reſpectively.

331. *Corol.* 2. Alſo this aberration $FT = \frac{mm}{\overline{m-n}, n} \times \frac{AP^2}{2EC} = \frac{mm}{\overline{m-n}|^2} \times$
d Euc. VI. 13. $\frac{AP^2}{2CF}$. For $PC = \frac{AP^2}{2EC}$ very nearly[d], and $EC = \frac{m-n}{n} \times CF$[e].
e Art. 224.

332. *Corol.* 3. Let the refracted ray *ATG* produced, cut the line *FG*, perpendicular to the axis, in *G*, and the lateral aberration $FG = \frac{mm}{nn} \times \frac{AP^3}{2EC^2} = \frac{mm}{\overline{m-n}|^2} \times \frac{AP^3}{2CF^2}$. For $FG : TF :: AP : TP$ or CF or $\frac{n}{m-n} \times CE$.

333. *Corol.* 4. When the ſemidiameter of the convexity or the focal diſtance is given, the longitudinal aberrations are as the ſquares, and the lateral aberrations as the cubes, of the linear apertures of a plano-convex lens.

PROPO-

PROPOSITION III.

334. *When parallel rays* QA, EC *are reflected from a ſpherical concave* ACB *whoſe center is* E *and whoſe aperture* ACB *is but ſmall, the longitudinal aberration* TF *of the extream ray* AT *from the geometrical focus* F, *is equal to half the verſed ſine* CP *of the ſemiaperture* AC *very nearly.* Fig. 380.

In fig. 379. imagine EM, the ſine of refraction to be diminiſhed to nothing and then to become negative and equal to EN the ſine of incidence, and the refraction of the ray to be changed to reflection as in fig. 380; and by the former propoſition it will be as $TF : CP :: mm : \overline{-m-n}, n$ $:: nn : -2nn :: 1 : -2$.

But the particular proof is this. By the laſt lemma the verſed ſine CP nearly equals $\frac{1}{2}$ the verſed ſine PD of the arch AD whoſe center is T and ſemidiameter TA or TE or $\frac{1}{2}$ the ſemidiameter of the arch AC[a] very nearly. But $2TF = 2TE - 2EF = ED - EC = CD$ exactly or CP nearly. Therefore $TF = \frac{1}{2}CP$ nearly. Fig. 380.

335. *Corol.* 1. We had $2TF = CD$ exactly; which is the exceſs of the ſecant ED of the arch AC above its radius EA. For joyning AD the angle DAE in the ſemicircle DAE is a right one.

336. *Corol.* 2. The longitudinal aberration $TF = \frac{AP^2}{4CE}$. For $CP = \frac{AP^2}{2CE}$ nearly[b].

337. *Corol.* 3. The lateral aberration $FG = \frac{AP^3}{2CE^2}$. For $FG : FT$ $:: AP : PT$ or $\frac{1}{2}CE$ nearly.

338. *Corol.* 4. When the diameter of the concave, or its focal diſtance, is given, the longitudinal aberrations are as the ſquares and the lateral ones as the cubes of the diameters of the apertures.

PROPOSITION IV.

339. *When parallel rays of any one ſort are refracted by a plano-convex object-glaſs, or when rays of all ſorts are reflected by a ſpherical concave, the diameter of each circle of aberrations cauſed by the ſphericalneſs of the figures, is equal to* $\frac{1}{2}$ *the lateral aberration of the extream ray in each; and therefore is given by the former propoſitions.*

Let $\alpha Y\tau$ be any refracted or reflected ray cutting the axis ECT in τ, and the extream ray ATG, that comes from the contrary ſide of the axis, in Y. Draw YX perpendicular to the axis, and ſuppoſing the line ATG immoveable, as the point of incidence α moves from the vertex C, the perpendicular XY will firſt increaſe, becauſe the angle $C\tau\alpha$ continually increaſes, Fig. 381, 382.

[a] Art. 205.

[b] Euc. VI. 13.

creaſes, and afterwards will decreaſe, becauſe the line $T\tau$ continually decreaſes; and when XY is the greateſt, it is evident that all the rays, incident upon the ſame ſide of the axis as it ſelf, will paſs through it. To find its greateſt quantity, let the incident ray $q\alpha$ cut the chord APB in β, and ſuppoſing the variable aperture $P\beta = v$, the variable $TX = x$ and the given lines $PA = a$, $PT = f$, $TF = b$; by corr. 4. prop. 2 and 3, the aberration $F\tau$ is to the aberration FT (b) as $\varpi\alpha^2$ or $P\beta^2$ (vv) to PA^2 (aa). Wherefore $F\tau = \frac{vv}{aa}b$ and thence $TF - F\tau = T\tau = \frac{b}{aa} \times \overline{aa - vv}$. Again PT (f) : PA (a) :: TX (x) : $XY = \frac{ax}{f}$; alſo $\varpi\alpha$ (v) : $\varpi\tau$ or PT (f) :: $XY\left(\frac{ax}{f}\right)$: $X\tau = \frac{ax}{v}$. Hence again $T\tau$, or $X\tau + XT = \frac{ax}{v} + x = \frac{b}{aa} \times \overline{aa - vv}$ found before; or $\frac{x}{v} \times \overline{a + v} = \frac{b}{aa} \times \overline{a + v} \times \overline{a - v}$. Whence $x = \frac{b}{aa}v \times \overline{a - v}$, and therefore x or TX is the greateſt poſſible when the rectangle $v \times \overline{a - v}$, or $P\beta \times \beta B$ is greateſt, that is when its ſides $P\beta$,

a Euc. VI. 13. βB are equal[a], or when $v = \frac{1}{2}a$. Subſtitute this value for v in the laſt equation and it gives the greateſt value of $x = \frac{1}{4}b$ or the greateſt $TX = \frac{1}{4}TF$, and therefore the greateſt $XY = \frac{1}{4}FG$, becauſe $TX : XY :: TF : FG$, and this XY turned about the axis PX deſcribes the circle of aberrations through which all the rays falling upon AB will juſt paſs. Q. E. D.

PROPOSITION V.

340. *The circle of aberrations cauſed by the ſphericalneſs of the figure of the object-glaſs of a teleſcope, compared with the circle of aberrations cauſed by the unequal refrangibility of rays, is altogether inconſiderable.*

Newt. Opt. p. 85. For if the object-glaſs be plano-convex and the plane ſide be turned towards the object, and the diameter of a ſphere whereof this glaſs is a ſegment be called D, and the ſemidiameter of the aperture of the glaſs be called S, and the ſine of incidence out of glaſs into air be to the ſine of refraction as n to m; the rays which come parallel to the axis of the glaſs ſhall in the place where the image of the object is moſt diſtinctly made, be ſcattered all over a little circle whoſe diameter is $\frac{mm}{nn} \times \frac{S^3}{DD}$ very nearly, if they were all equally refrangible by article 339 and 332. As for inſtance if the ſine of incidence n be to the ſine of refraction m as 20 to 31, and if D, the diameter of the ſphere to which the convex ſide of the glaſs is ground, be 100 foot or 1200 inches, and conſequently the teleſ-

b Art. 224. cope about 100 foot long[b], and S the ſemidiameter of the aperture be 2 inches, the diameter of this circle of aberrations, that is $\frac{mm}{nn} \times \frac{S^3}{DD}$, will be $\frac{31\times 31\times 8}{20\times 20\times 1200\times 1200}$ or $\frac{961}{72000000}$ parts of an inch. But the diameter of the little circle through which theſe rays are ſcattered by unequal refrangibility, will be about the 55th part of the breadth of the aperture of the object-glaſs[a], which is here 4 inches. And therefore the aberration ariſing from the ſpherical figure of the glaſs, is to the aberration ariſing from the different refrangibility, as $\frac{961}{72000000}$ to $\frac{4}{55}$, that is as 1 to 5449; and therefore being in compariſon ſo very little, deſerves not to be conſidered in the theory of teleſcopes. If we ſuppoſe the little circle of aberrations ariſing from unequal refrangibility, to be 250 times narrower than the circular aperture of the object-glaſs, it would contain all the orange and yellow, and would permit the other fainter and darker colours to paſs by it[b], which perhaps may ſcarce affect the ſenſe; yet even in this caſe the aberration cauſed by the ſpherical figure, would be to the aberration cauſed by the unequal refrangibility, in a 100 foot teleſcope, but as $\frac{961}{72000000}$ to $\frac{4}{250}$ or only as 1 to 1200, which ſufficiently proves the propoſition. *Q. E. D.*

341. *Corol.* 1. If the focal diſtances and apertures of a reflecting concave and a plano-convex glaſs be both the ſame, the diameter of the circle of aberrations, cauſed by their figures, will be above 30 times leſs in the reflecter than in the refracter. For theſe diameters are $\frac{AP^3}{16CF^2}$ and $\frac{mm}{\overline{m-n}|^2}\times\frac{AP^3}{4CF^2}$ by art. 339, 337 and 332; which are as $\frac{1}{4}$ to $\frac{mm}{\overline{m-n}|^2}$ or $\frac{31\times 31}{11\times 11}$. Hence if the length of each teleſcope be 100 foot, the lateral aberrations in the reflecter would be 30 × 5449 or 163470 times leſs than the lateral aberrations cauſed by unequal refrangibility in the refracter.

342. *Corol.* 2. The number of pencils, ſome of whoſe rays are mixed together in every point of a confuſed picture, is as the area of the circle of aberrations of the rays in any one pencil; and conſequently the mixture of the rays of different pencils, cauſed by the ſphericalneſs of the figure of an object-glaſs, if they were all alike refrangible, would be to their mixture cauſed by their unequal refrangibility, as 1 to 5449 × 5449 or 29691601 in the preſent inſtance. For conceiving any point in the confuſed picture to be a center of a circle of aberrations, it is manifeſt that all other equal circles of aberrations, whoſe centers fall upon the firſt mentioned circle will cover its center; that is ſome rays of as many pencils will be mixed in this center as there are points in the circle it ſelf; or, which is the ſame thing, the number of pencils mixed in this center is as the area of the circle of aberrations.

[a] Art. 323.

[b] Art. 326. Newt. Opt. p. 88.

CHAPTER VII.

A refracting or reflecting teleſcope being given, whoſe aperture and eye-glaſs are adjuſted by experience, to determine the length, aperture and eye-glaſs of another teleſcope, through which an object ſhall appear as bright and diſtinct as in the given one, and magnified as much as ſhall be required.

PROPOSITION I.

343. IN *all ſorts of teleſcopes and double microſcopes, the apparent indiſtinctneſs of a given object, is as the area of a circle of aberrations in the focus of the object-glaſs directly, and as the ſquare of the focal diſtance of the eye-glaſs inverſely.*

For in viſion with the naked eye or with glaſſes, the apparent indiſtinctneſs of a given object, is as the area of a circle of aberrations in its picture painted upon the retina. Becauſe any one ſenſible point of the retina, being the center of a circle of aberrations, will at once be affected by a mixture of the rays of as many diſtinct pencils, as there are ſenſible points in the area of that circle [a]; and ſo will at once convey to the mind a mixt or confuſed ſenſation of the ſame number of viſible points in the object, from whence thoſe pencils flowed; and this number of points is as the magnitude of the area of a circle of aberrations, whatever be the magnitude of a ſenſible point of the retina. Now in viſion with teleſcopes, the diameter of a circle of aberrations in the picture upon the retina, is as the apparent magnitude of the diameter of the correſponding circle of aberrations in the common focus of the glaſſes [b], that is as the angle ſubtended by this diameter at the center of the eye-glaſs [c]; that is as the diameter it ſelf directly, and the focal diſtance of the eye-glaſs inverſely [d]. And ſo the area of that circle of aberrations upon the retina, is as the area of the correſponding circle of aberrations in the focus of the object-glaſs directly, and as the ſquare of the focal diſtance of the eye-glaſs inverſely [e].

344. *Corol.* In all ſorts of teleſcopes and double microſcopes a given object appears equally diſtinct, when the focal diſtances of the eye-glaſſes are as the diameters of the circles of aberrations in the focus of the object-glaſſes.

345. The alteration in the confuſion which may ariſe from aberrations cauſed by the eye-glaſſes, is not here regarded, as being inconſiderable. We only conſider the confuſion of thoſe points in the image which lye very near the axis of the teleſcope, as of the point q in fig. 181. Now if this point was perfectly diſtinct the rays going from it would emerge from

the

[a] Art. 342.
[b] Art. 91.
[c] Art. 120.
[d] Art. 222.
[e] Euc. XII. 2.

the eye-glaſs in parallel lines without ſenſible error; becauſe the breadth of this cylinder of rays is exceeding ſmall compared to the breadth of the eye-glaſs, being in proportion to the breadth of the aperture of the object-glaſs as their focal diſtances; and the refractions at ſo ſmall a diſtance from the axis are ſufficiently true and regular. It is the largeneſs of the aperture of the object-glaſs and of its focal diſtance, which cauſes the irregularity in its refractions. Add to this that the differently refrangible rays cannot be ſeparated ſenſibly in going ſo ſhort a diſtance as between the eye-glaſs and the eye. Beſides this we find by experience that objects and images diſtinct in themſelves, appear ſufficiently diſtinct through very ſmall eye-glaſſes when their apertures are ſmall. This remark will be demonſtrated more diſtinctly in the 11th chapter, where the ſame ſubject is handled more fully.

PROPOSITION II.

346. *In refracting teleſcopes the apparent indiſtinctneſs of a given object, is directly as the area of the aperture of the object-glaſs, and inverſely as the ſquare of the focal diſtance of the eye-glaſs.*

This appears from prop. 1, becauſe the area of the circle of aberrations at the focus of the object-glaſs is as the area of its aperture[a]; and becauſe the aberrations ariſing from the eye-glaſs[b], and from the ſphericalneſs of the figure of them both are inconſiderable[c].

[a] Art. 324.
[b] Art. 345.
[c] Art. 340.

347. *Corol.* In refracting teleſcopes a given object appears equally diſtinct, when the diameters of the apertures of their object-glaſſes, are as the focal diſtances of their eye-glaſſes.

PROPOSITION III.

348. *In all ſorts of teleſcopes and double microſcopes the apparent brightneſs of a given object is as the ſquare of their linear apertures directly and as the ſquare of their linear amplifications inverſely.*

For if the ſquares of the linear amplifications, that is if the areas of the pictures upon the retina were the ſame, their brightneſs would be as the quantities of light coming through the areas of the apertures, that is as the ſquares of the linear apertures; and if the apertures or quantities of light were the ſame, the brightneſs of the pictures would be as their areas inverſely or as the ſquares of the linear amplifications inverſely. Therefore when neither the apertures nor the amplifications are the ſame, the brightneſs is as the ſquare of the linear apertures directly, and as the ſquare of the linear amplifications inverſely. *Q. E. D.*

349. *Corol.* 1. Hence in refracting and reflecting teleſcopes a given object appears equally bright, when their linear apertures are as their linear amplifications, that is as the focal diſtances of the object-glaſſes directly and as the focal diſtances of the eye-glaſſes inverſely.

350. *Corol.* 2. If the breadth of the aperture of a given object-glaſs and the focal diſtance of the eye-glaſs be each increaſed in any given ratio, the
a Art. 347. diſtinctneſs will remain the ſame as before[a]; and the linear amplification
b Art. 120. will be diminiſhed in the ſame ratio[b]; but the apparent brightneſs will be increaſed in a ratio quadruplicate of the former ratio by this propoſition; and on the contrary.

Dioptr. p. 215. 351. *Hugens* obſerves that the ſame degrees of diſtinctneſs here demonſtrated do not exactly agree with experience, as he found by looking at the ſame object through different teleſcopes, or through the ſame teleſcope with different apertures; and that through the larger aperture the object appeared not quite ſo diſtinct as through the ſmaller. He found alſo that in viewing objects of different brightneſs through the ſame aperture, the apparent indiſtinctneſs of the brighter object was a little greater than that of the duller: and therefore the aperture adjuſted for the duller planets may be ſomewhat larger than for the brighter.

PROPOSITION IV.

352. *In reflecting teleſcopes the apparent indiſtinctneſs of a given object is as the ſixth power of the diameter of the aperture of the object-metal directly, and as the fourth power of its focal diſtance inverſely, and alſo as the ſquare of the focal diſtance of the eye-glaſs inverſely.*

For the area of a circle of aberations in the focus of the object-metal is as the ſixth power of its linear aperture directly and as the fourth power
a Art. 339. of its focal diſtance inverſely[a]; and therefore the apparent indiſtinctneſs
337. of the object, is as the ſixth power of the linear aperture directly, as the fourth power of the focal diſtance of the object-metal inverſely, and as the
b Art. 343. ſquare of the focal diſtance of the eye-glaſs inverſely[b]. *Q. E. D.*

353. *Corol.* In reflecting teleſcopes a given object appears equally diſtinct when the cubes of the linear apertures of the object-metals, are as the ſolids whoſe baſes are the ſquares of the focal diſtances of the object-metals, and heights are the focal diſtances of the eye-glaſſes: or when the focal diſtances of the eye-glaſſes are as the cubes of the linear apertures of the object-metals, applyed to the ſquares of their focal diſtances.

PROPOSITION V.

354. *In refracting teleſcopes of various lengths a given object will appear equally bright and equally diſtinct, when their linear apertures and focal diſtances of their eye-glaſſes are ſeverally in a ſubduplicate ratio of their lengths or focal diſtances of their object-glaſſes: and then alſo their linear amplifications will be in a ſubduplicate ratio of their lengths.*

For to ſhew the object equally bright, the rectangle under the linear aperture and the focal diſtance of the eye-glaſs muſt be as the length of the teleſcope [a], and to ſhew it equally diſtinct the linear aperture muſt be as the focal diſtance of the eye-glaſs [b]; and therefore to perform both things together, the ſquare of the linear aperture, and alſo the ſquare of the focal diſtance of the eye-glaſs, muſt be ſeverally (as the rectangle under each, or) as the length of the teleſcope; and conſequently the linear aperture, and alſo the focal diſtance of the eye-glaſs, as the ſquare root of that length. Now the linear amplification was as the linear aperture [c], or by this demonſtration, as the ſquare root of the length of the teleſcope. *Q. E. D.*

355. *Hugens*'s ſtandard teleſcope 30 foot long, or 360 inches, bears an aperture whoſe breadth is 3 inches, and an eye-glaſs whoſe focal diſtance is 3 inches and 3 tenths. From whence he has given us the following table of apertures and eye-glaſſes for other teleſcopes [d], computed by the following rule.

Multiply the number of feet in the focal diſtance of any propoſed object-glaſs by 3000, and the ſquare root of the product will give the breadth of its aperture in hundredth parts of an inch. And the ſame breadth of the aperture, increaſed by a tenth part of it ſelf, gives the focal diſtance of the eye-glaſs in hundredth parts of an inch. And the magnifying powers are as the breadths of the apertures.

For ſince the ſtandard teleſcope has 30 foot focal diſtance of its object-glaſs, put F for the number of feet in any other focal diſtance, and ſay by the propoſition as $\sqrt{30}$ to $\sqrt{F}$, ſo is the ſtandard aperture 3 inches, or 300 centeſimals or $\sqrt{300 \times 300}$, to the aperture ſought; which therefore is $\sqrt{3000F}$ in centeſimals of an inch. The focal diſtance of the eye-glaſs of the ſtandard teleſcope is $3\frac{3}{10}$ inches, that is a tenth part more than the breadth of the aperture of the object-glaſs; conſequently the focal diſtance of the new eye-glaſs muſt be a tenth part more than the linear aperture of the new object-glaſs, by the laſt propoſition.

356. He alſo adds the following directions how to ſuit theſe teleſcopes to all ſorts of objects ſeen either by day or by night. They are proportioned in the following table for aſtronomical obſervations, and therefore will require more light when uſed in the day time. For when the eye is dazled with the brightneſs of the day, objects will appear through them but obſcure, which in the night are ſufficiently bright. Therefore (ſays *Hugens*) when I uſed theſe teleſcopes to obſerve objects by day-light, by experience I found it requiſite to change the eye-glaſſes for others whoſe focal diſtances were double the former. By this means the apparent brightneſs became quadruple, becauſe the ſurfaces of the images in the bottom of the eye were diminiſhed in the ſame proportion [e]. For as the aperture remains unaltered, ſo does the quantity of light, and therefore it illuminates a leſſer ſpace ſo much the more. Now if the aperture was increaſed

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[a] Art. 349.

[b] Art. 347.

[c] Art. 347.

[d] Art. 364.

[e] Art. 120.

Diop. p. 210.

Dioptr p. 215.

without changing the eye-glaſs, the brightneſs would be increaſed too, but then the miſt ariſing from greater aberrations would alſo be greater; and therefore this remedy muſt not be uſed.

357. But one may aſk this queſtion, ſince by ſubſtituting an eye-glaſs of a longer focal diſtance, the apparent indiſtinctneſs hitherto examined is diminiſhed, why may not the aperture of the object-glaſs be ſo far increaſed, till the ſame degree of indiſtinctneſs returns again as belongs to a teleſcope regulated by the table? For from hence more light is gained and the diſtinctneſs is not altered[a]. The anſwer is this, which I hinted before[b], that the miſt ariſing from *Newton*'s aberration, though the ſame in quantity, becomes more ſenſible in proportion to the brightneſs of the image. For the brightneſs of the miſt increaſes at the ſame time. And we find by experience, that as ſoon as the apertures of thoſe day-light teleſcopes are increaſed, the miſt ariſing from the aberrations of a brighter object begins to be troubleſome. The apertures therefore muſt not be altered.

358. Again one may aſk, if a teleſcope fitted for Saturn be applyed to the Moon, which is 100 times brighter (I mean in each equal parts, though not in the whole, as being 10 times nearer to the Sun;) one may aſk I ſay whether the breadth of the aperture and the focal diſtance of the eye-glaſs may not both be leſſened in the ſame proportion, to make the regions of the moon no brighter than thoſe of ſaturn, but much greater in appearance than before. For inſtance, in a 30 foot teleſcope, if 3 inches, the breadth of the aperture be reduced to $\sqrt{\frac{9}{10}}$ of an inch, which is ſomewhat leſs than ($\sqrt{\frac{9}{9}}$ or) a third part of the former, and alſo the focal diſtance of the eye-glaſs be ſhortened in the ſame proportion; the proportion of the apparent brightneſs in theſe two teleſcopes, the object being the ſame, would be quadruplicate of 3 to $\sqrt{\frac{9}{10}}$[c] that is as 100 to 1; and ſince the regions in the moon are 100 times brighter in themſelves than thoſe in ſaturn, the moon would appear in the darker teleſcope juſt as bright as ſaturn did in the lighter. But the apparent indiſtinctneſs hitherto conſidered would alſo be the ſame in both[d], and the amplification of the moon would be greater than that of ſaturn in the ratio of 3 to $\sqrt{\frac{9}{10}}$[e], which is more than triple. So that this reduction of the aperture and eye-glaſs ſeems very advantageous; but in reality it is quite otherwiſe; and that for two reaſons. Firſt becauſe the minute parts of the moon may be better diſcerned when all the light remains in the teleſcope, than when it is reduced to an 100th part, though not in the ſame proportion. The other reaſon is that when the aperture is too much contracted, the out-lines that circumſcribe the pictures in the eye become confuſed; which is carefully to be minded, and alſo what are the limits of this confuſion. This is certain that as the aperture is contracted, the ſlender pencils or cylinders of rays, that emerge from the eye-glaſs into the eye, are alſo contracted

[a] Art. 350.

[b] Art. 351.

[c] Art. 350.

[d] Art. 350.

[e] Art. 120.

ed in the ſame proportion. Now if the breadth of one of theſe pencils be leſs than $\frac{1}{5}$ or $\frac{1}{6}$ of a line, that is leſs than $\frac{1}{60}$ or $\frac{1}{72}$ part of an inch, the out-lines of the pictures are ſpoiled, for ſome unknown reaſon in the make of the eye, whether in the choroid, or in the retina, or in the humors it is uncertain. For by looking through an hole, in a thin plate, narrower than $\frac{1}{5}$ or $\frac{1}{6}$ of a line, the edges of objects begin to appear confuſed and ſo much the more as the hole is made narrower. Now it is eaſy to ſhew in the laſt mentioned teleſcope that the cylinder of rays is too ſlender. For by adding $\frac{1}{10}$ of the aperture to it ſelf[a], the focal diſtance of the eye-glaſs becomes $\sqrt{\frac{9}{10}} + \frac{1}{10}\sqrt{\frac{9}{10}}$, that is $\frac{11}{10}\sqrt{\frac{9}{10}}$ of an inch; and by ſimilar triangles ſubtended at the common focus q by the aperture and cylinder ſought[b], it is as the focal diſtance of the object-glaſs, to the focal diſtance of the eye-glaſs, ſo the breadth of the aperture, to the breadth of the cylinder; that is as 30 feet or 360 inches to $\frac{11}{10}\sqrt{\frac{9}{10}}$ inches, ſo is $\sqrt{\frac{9}{10}}$ of an inch to $\frac{11}{4000}$ inch or almoſt $\frac{1}{30}$ of a line; which is much leſs than $\frac{1}{6}$. But in the teleſcope regulated in the table, it is as 360 to $3\frac{1}{10}$ ſo 3 to $\frac{11}{400}$ of an inch or almoſt $\frac{1}{3}$ of a line for the breadth of that cylinder; which can poſſibly do no harm. Hence we learn that the breadth of the aperture and focal diſtance of the eye-glaſs cannot be contracted much more than $\frac{1}{3}$ of themſelves; for even then the breadth of the cylinder at the eye will not much exceed $\frac{1}{5}$ of a line. The ſame is to be underſtood of teleſcopes of all lengths regulated as in the table, the breadth of the cylinder being the ſame in all. For by the proportion juſt mentioned it equals the breadth of the aperture multiplied into the focal diſtance of the eye-glaſs and divided by the focal diſtance of the object-glaſs, and conſequently it is proportionable to the linear aperture directly and the linear amplification inverſely; which two ratios muſt compound a ratio of equality to preſerve the ſame apparent brightneſs, by art. 349.

359. Hence though we transferred one of theſe teleſcopes from Saturn to Venus which is 225 times brighter, being 15 times nearer to the Sun, yet the breadth of the aperture muſt not be contracted above $\frac{1}{3}$ part of the whole; and if too much light ſtill remains, it muſt be diminiſhed by darkening the eye-glaſs with the ſmoak of a candle. For a greater contraction of the aperture is hurtful for another reaſon, that all the little bubbles and veins in the eye-glaſs become more conſpicuous by intercepting the whole or a greater part of thoſe little cylinders above mentioned, and conſequently the particles of the object they came from.

360. Upon the whole I conclude we may lengthen our teleſcopes at pleaſure, according to the laws of the table, with good ſucceſs; ſince not only the brightneſs and diſtinctneſs remain unaltered, but alſo the breadth of the pencils that enter the eye. Laſtly to obſerve exceeding ſmall ſtars and eſpecially the Satellites of Jupiter and Saturn, the beſt way is to increaſe very much both the aperture and focal diſtance of the eye-glaſs.

[a] Art. 355.

[b] Fig. 181.

For ſince they appear like points even through the teleſcope, there is nothing gained by endeavouring to increaſe their diameters; but their brightneſs muſt be increaſed as much as poſſible; and this is chiefly done by increaſing the aperture. By doubling its breadth, the light received into it becomes quadruple, and then by doubling alſo the focal diſtance of the eye-glaſs, the diſtinctneſs returns to the ſame as at firſt [a]. But ſtill the brightneſs will not become 16 times greater, according to cor. 2. prop. 3, but only 4 times; becauſe as I ſaid the picture of the ſtar upon the retina is but a ſenſible point, whoſe brightneſs cannot therefore be increaſed by a diminution of its breadth, but only by an addition of new light. The caſe is different when we view the moon and primary planets through the ſame teleſcope, whoſe ſeveral parts receive 16 times more light than before. Thus by widening the apertures we very much increaſe the power of the teleſcope for finding out ſmall ſtars and the ſatellites of Saturn, ſo that perhaps with a 30 foot glaſs, whoſe aperture is 6 inches or double the uſual one, as much may be done as with another of 120 foot whoſe aperture by the table is alſo 6 inches. So far from *Hugens.*

[a] Art. 347.

PROPOSITION VI.

361. *In reflecting teleſcopes of various lengths a given object will appear equally bright and equally diſtinct, when their linear apertures and alſo their linear amplifications are as the ſquare-ſquare roots of the cubes of their lengths: and conſequently when the focal diſtances of their eye-glaſſes are alſo as the ſquare-ſquare roots of their lengths.*

Put A for the linear aperture of the reflecting concave, L for its focal diſtance or the length of the teleſcope, F for the focal diſtance of the eye-glaſs; and when the diſtinctneſs is given A^{3} is as FLL^{*}; and when the brightneſs is given the amplification or $\frac{L}{F}$ is as A^{*}, that is F is as $\frac{L}{A}$. Therefore when the diſtinctneſs and brightneſs are both given, A^{3} is as $\frac{L^{3}}{A}$; or A^{4} as L^{3}; or A as $\sqrt[4]{L^{3}}$. The amplification $\frac{L}{F}$ was as A, that is as $\sqrt[4]{L^{3}}$; and therefore F is as $\frac{\sqrt[4]{L^{4}}}{\sqrt[4]{L^{3}}}$ or $\sqrt[4]{L}$. Q.E.D.

* Art. 353.

* Art. 349.

362. In the reflecting teleſcope made and deſcribed by *John Hadley* Eſq; F.R.S. in the Philoſophical Tranſactions N°. 376 and 378, $L = 62\frac{1}{2}$ inches, $F = \frac{1}{3}$ or $\frac{3}{10}$ or $\frac{11}{40}$ of an inch. For he uſes 3 eye-glaſſes and as many apertures for the reflecter whoſe breadths are $4\frac{1}{2}$, 5, $5\frac{1}{2}$ inches. Hence the linear amplifications or $\frac{L}{F}$ are $187\frac{1}{2}$, $208\frac{1}{3}$, $227\frac{3}{11}$ reſpectively. Ta-

king the middle eye-glaſs and aperture for a ſtandard I computed the following table for teleſcopes of other lengths by this Rule. Call the number of inches in the length of any teleſcope L, and the focal diſtance of its eye-glaſs will be equal to $60\sqrt[4]{10\,L}$ in thouſandth parts of an inch. The quotient of L divided by $60\sqrt[4]{10\,L}$ or F gives the amplification[a], which multiplied by 24 will always give the linear aperture in thouſandth parts of an inch. For by the propoſition $\sqrt[4]{L}$ is as F; that is $\sqrt[4]{62\frac{1}{2}}$ or $\sqrt[4]{\frac{125}{2}}$ or $\sqrt[4]{\frac{625}{10}}$ or $5\sqrt[4]{\frac{1}{10}}$ is to $\sqrt[4]{L}$ as $\frac{3}{10}$ or 300 milleſimals in the given eye-glaſs, to the milleſimals in the correſpondent eye-glaſs or in $F = 60\sqrt[4]{10\,L}$. And the aperture being as the amplification by the propoſition, ſay, as the amplification given or $208\frac{1}{3}$ is to $\frac{L}{F}$, the amplification found, ſo is 5 inches, the aperture given, to the aperture ſought $= \frac{5}{208\frac{1}{3}} \times \frac{L}{F} = \frac{24}{1000} \times \frac{L}{F}$ inches.

363. Were it not for the unequal refrangibility of rays, refracting teleſcopes, though not ſo ſhort as theſe[b], would alſo be proportioned by this rule[c]: which not agreeing with experience, ſhews again that the aberration ariſing from the ſpherical figure are inconſiderable in compariſon to the other aberrations ariſing from the unequal refrangibility of the rays.

[a] Art. 125.

[b] Art. 341.

[c] Art. 333, 338.

364. **REFRACTING TELESCOPES**

Length of the teleſcope or focal diſt. of the object-glaſs. Feet.	Linear aperture of the object-glaſs. Inch. & Dec.	Focal diſt. of the eye-glaſs. Inch. & Dec.	Linear amplification, or magnifying power.
1	0.55	0.61	20
2	0.77	0.85	28
3	0.95	1.05	34
4	1.09	1.20	40
5	1.23	1.35	44
6	1.34	1.47	49
7	1.45	1.60	53
8	1.55	1.71	56
9	1.64	1.80	60
10	1.73	1.90	63
13	1.97	2.17	72
15	2.12	2.32	77
20	2.45	2.70	89
25	2.74	3.01	100
30	3.00	3.30	109
35	3.24	3.56	118
40	3.46	3.81	126
45	3.67	4.04	133
50	3.87	4.26	141
55	4.06	4.47	148
60	4.24	4.66	154
70	4.58	5.04	166
80	4.90	5.39	178
90	5.20	5.72	189
100	5.48	6.03	199
120	6.00	6.60	218
140	6.48	7.13	235
160	6.93	7.62	252
180	7.35	8.09	267
200	7.75	8.53	281
220	8.12	8.93	295
240	8.48	8.83	308
260	8.83	9.71	321
280	9.16	10.08	333
300	9.49	10.44	345
400	10.95	12.05	398
500	12.25	13.47	445
600	13.42	14.76	488

REFLECTING TELESCOPES

Length of the teleſcope or focal diſt. of the concave Feet.	Focal diſt. of the eye-glaſs. Milleſ. Inch.	Linear amplification, or magnifying power.	Linear aperture of the concave metal. Milleſ. Inch.
½	0.167	36	0.864
1	0.199	60	1.440
2	0.236	102	2.448
3	0.261	138	3.312
4	0.281	171	4.104
5	0.297	202	4.848
6	0.311	232	5.568
7	0.323	260	6.240
8	0.334	287	6.888
9	0.344	314	7.536
10	0.353	340	8.160
11	0.362	365	8.760
12	0.367	390	9.360
13	0.377	414	9.936
14	0.384	437	10.488
15	0.391	460	11.040
16	0.397	483	11.592
17	0.403	506	12.143

365. Theſe proportions, in *Hugens*'s table for refracting teleſcopes, are meaſured by the Rheinland foot which is to the Engliſh foot as 139 to 135; ſo that taking their lengths of as many engliſh feet, their apertures and eye-glaſſes and linear amplifications ſhould be ſeverally diminiſhed in the ſubduplicate ratio of 139 to 135 by art. 354. that is nearly in the ratio of 139 to 137 or about $\frac{1}{69}$ or $\frac{1}{70}$ part of the whole.

CHAPTER VIII.

Contains general properties of focuſes and images, belonging to the eye and to any number of mediums: with general conſtructions, ſhewing the variations of the apparent diſtance of an object, and of the real diſtance of its laſt image, from the eye, cauſed by a direct motion of the eye, object or mediums.

PROPOSITION I.

HAVING *the diameters and poſitions of two ſpherical ſurfaces, which intercede three given mediums, and ſuppoſing the incident rays in either of the outward mediums to be parallel, and very near to the common axis of the ſurfaces, it is propoſed to find their focus after both refractions.* Fig. 383.

366. In the common axis AC of the ſurfaces AB, CD, let a and d be the focuſes of rays, which before they emerged by refractions at AB and CD into the outward mediums, went both ways parallel to the axis in the inward medium. Moreover let b and c be the focuſes of other rays, which before they emerged by refractions at AB and CD into the inner medium, came both ways parallel to the axis in the outward mediums. Theſe focuſes may be found by art. 224. Then ſay as $cb : bA :: Aa : aI$; and placing aI the contrary way from a to that of bc from b, the point I ſhall be the focus of the rays after both refractions, which came parallel from without upon the ſurface CD[a]: becauſe c is their focus after their firſt refraction. In like manner ſay as $bc : cC :: Cd : dK$, and placing dK the contrary way from d to that of cb from c, the point K ſhall be the focus of other rays after both refractions, which came parallel from without upon the ſurface AB.

All the figures are adapted to mediums whoſe denſities are continually greater as they lye in order from the left hand to the right; but the demonſtrations ſerve for any irregular order of denſities.

PROPOSITION II.

The focus of incident rays being given, it is propoſed to find their focus after their refractions at two ſpherical ſurfaces, which intercede given mediums.

367. In the common axis AC of the given ſurfaces AB, CD, let a and d be the focuſes of rays, which before they emerged by refractions into the outward mediums, went both ways parallel to the axis in the inner medium. Moreover let I and K be the focuſes of other rays, which before Fig. 384.

[a] Art. 237.

fore their refractions through both the ſurfaces, came both ways parallel to the axis in the outward mediums. Then let P be the given focus of the incident rays, I and a the focuſes of thoſe rays which came the contrary way to the incident rays, and ſay as PI to Ia ſo dK to KR; and placing KR the contrary way from K to that of IP from I, the point R ſhall be their focus after both the refractions.

Fig. 385. For by ſaying as $Pa : aA :: Ab : bQ$ and by placing bQ the contrary way from b to that of aP from a, the point Q is their focus after the firſt refraction at AB[a]. The ſame point Q being the focus of incident rays upon the ſurface CD, ſay again as $Qc : cC :: Cd : dR$; and placing dR the contrary way from d to that of cQ from c, the point R is their focus after both refractions. Now by the firſt of theſe proportions, and of thoſe in the foregoing propoſition, the rectangle $Pa \times bQ = (aA \times Ab =)\ bc \times Ia$; and by the ſecond proportions in this and in the foregoing propoſition, the rectangle $Qc \times dR = (cC \times Cd =)\ bc \times dK$; then by reſolving the two former rectangles into the proportion of their ſides, it is as $Ia : Pa :: bQ : bc$ and disjointly (or conjointly) $PI : Pa :: (Qc : bc ::)\ dK : dR$ by reſolving the two latter rectangles; and disjointly (or conjointly) $PI : Ia :: dK : KR$.

368. *Corol.* The magnitude of the rectangle under PI, KR is invariable, being always equal to the given rectangle under Ia, dK; and conſequently KR is reciprocally as PI.

PROPOSITION III.

Having the diameters and poſitions of three ſpherical ſurfaces, which intercede four given mediums, if the incident rays in either of the outward mediums be parallel, and very near to the common axis of the ſurfaces, it is propoſed to find their focus after all the refractions.

Fig. 386. 369. Suppoſing the focuſes a, d and I, K to be determined by the foregoing propoſition for two contiguous ſurfaces, as AB and CD; let parallel rays falling on both ſides of the third ſurface EF have their focuſes, after refraction there only, at e and f; whereof e being the focus of incident rays upon the ſurface CD, ſay as $eK : Kd :: aI : IL$; and placing IL the contrary way from I to that of Ke from K, by the foregoing propoſition the point L is the focus of rays, after all the refractions, which came parallel from without upon the ſurface EF. Again, ſay as $Ke : eE :: Ef : fM$, and placing fM the contrary way from f to that of eK from e, the point M is the focus of rays after all the refractions which came parallel from without upon the ſurface AB[b].

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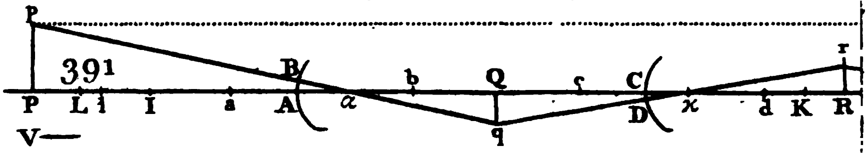
[a] Art. 237.

[b] Art. 237.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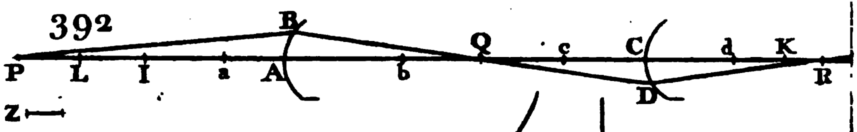
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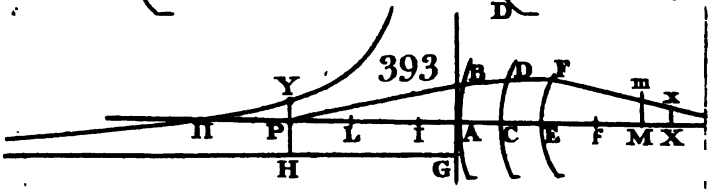
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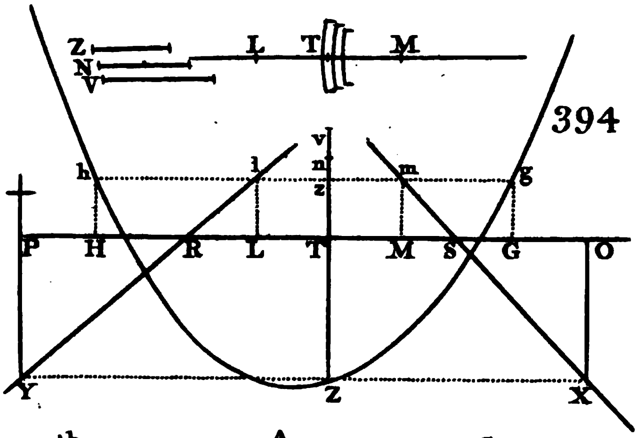
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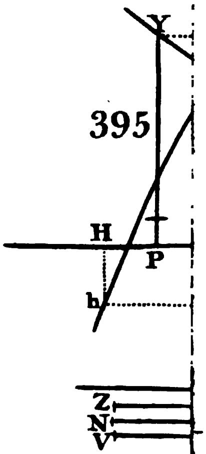
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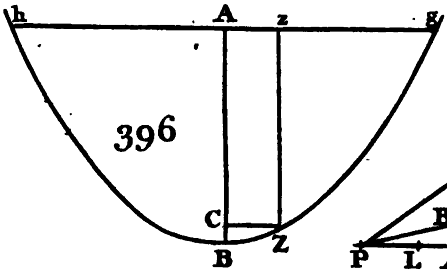
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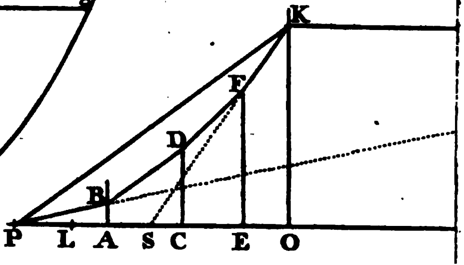
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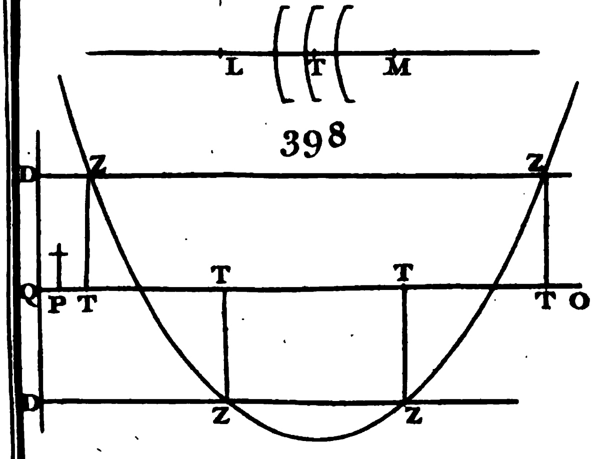
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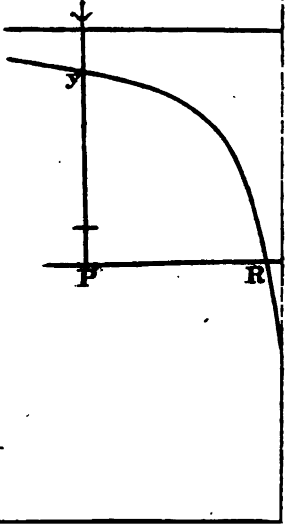
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PROPOSITION IV.

The focus of incident rays being given, it is propoſed to find their focus after refractions through any given number of ſpherical ſurfaces that intercede given mediums.

370. *Caſe* 1. In the common axis of three ſurfaces AB, CD, EF, let I Fig. 387. and f be the focuſes of rays, which before they emerged by refractions into the outward mediums, went both ways parallel to the axis in either of the inner mediums, as CE; moreover let L and M be the focuſes of other rays, which before their refractions through all the ſurfaces, came both ways parallel to the axis in the outward mediums; then let P be the given focus of the incident rays, L and I the focuſes of thoſe rays which came the contrary way to the incident rays, and ſay as $PL : LI :: fM : MS$; and placing MS the contrary way from M to that of LP from L, the point S ſhall be their focus after three refractions.

For by the ſecond propoſition their focus R, after two refractions at Fig. 388. the ſurfaces AB, CD, is given by ſaying as $PI : Ia :: dK : KR$; and ſince R is their focus when incident on the ſurface EF, ſay as $Re : eE :: Ef : fS$, and S is their focus after three refractions[a]. Now by the firſt of theſe proportions and of thoſe in the third propoſition, the rectangle $PI \times KR = (Ia \times dK =) Ke \times LI$. Likewiſe by the ſecond of theſe proportions and of thoſe in the third propoſition, the rectangle $Re \times fS = (eE \times Ef =) Ke \times fM$; and by reſolving the two former rectangles into the proportion of their ſides, it is as $LI : PI :: KR : Ke$, and disjointly (or conjointly) $PL : PI :: (Re : Ke ::) fM : fS$ by reſolving the latter rectangles; and disjointly (or conjointly) $PL : LI :: fM : MS$.

The rays whoſe focuſes are I and f were ſuppoſed to be parallel to the Fig. 389, 390. axis in the medium CE; now let a and t be the focuſes of other rays which are parallel to the axis in the other inner medium AC; and ſince $PL : LI :: fM : MS$; for the ſame reaſon when P comes to a and conſequently S to t, it is as $aL : LI :: fM : Mt$, therefore becauſe of the given rectangle $LI \times fM$, it is as $PL : La :: tM : MS$.

371. *Caſe* 2. Hence by the method of the third propoſition one may find the focus of parallel rays after refraction through four ſurfaces, and then the focus of inclined rays in the ſame manner as in the foregoing caſe, and ſo on. And by theſe propoſitions it is ſufficiently evident, that if L and M be the principal focuſes of the whole ſyſtem of ſurfaces, and I and f be the focuſes of other rays that go parallel in any one of the inner mediums, then $PL : LI :: fM : MS$.

372. *Corol.* 1. It is manifeſt by the analogy between the rules for finding the focuſes of rays refracted through a ſingle ſurface and a ſingle lens[b], that the rule in this propoſition for the focuſes of rays refracted through any number of ſurfaces, will ſerve alſo for any number of lenſes of any ſort placed

[a] Art. 237.

[b] Art. 236.

placed at A, C, D, &c. in one continued medium. And ſo it appears that the relation between the conjugate focuſes P, S, of a pencil of rays refracted through an infinite number of ſurfaces or lenſes, may be always expreſſed by a ſingle proportion; in like manner as that relation is expreſſed in a ſingle ſurface or a ſingle lens.

373. *Corol.* 2. Take a line N a middle proportional between the given lines LI, fM, or between La, tM, and it will alſo be a middle proportional between the variable lines PL, MS: and conſequently PL is reciprocally as MS, and they lye contrary ways from the principal focuſes L, M. And the emergent rays will flow from S if it lyes on the ſame ſide of the laſt ſurface as the incident rays, otherwiſe towards S. Becauſe the rays continue to go forwards from the laſt ſurface.

PROPOSITION V.

Fig. 391.

Having the ſemidiameter Pp *of a ſmall object, placed perpendicularly to the common axis of any number of refracting ſurfaces, which intercede given mediums; it is propoſed to find the ſemidiameter* Ss *of its laſt image.*

374. Things remaining as they were in the preceding propoſitions, take a line $V = Aa \times \frac{dK}{dC} \times \frac{fM}{fE}$ &c. as far as the number of ſurfaces permits; and Ss will be to Pp as the conſtant line V to PL.

For let α, $\varkappa$, η, &c. be the centers of the given ſurfaces A, C, E, &c; Qq, Rr, Ss, &c. the reſpective images formed by them. The object and theſe images are terminated by the lines $p\alpha q$, $q\varkappa r$, $r\eta s$, &c. Now by art. 238. it is as $Qb : QA :: Q\alpha : QP$, and disjointly (or conjointly) $Qb : bA :: Q\alpha : \alpha P :: Qq : Pp$, becauſe the triangles $Q\alpha q$, $P\alpha p$ are equiangular; and for the ſame reaſon that $Pp : Qq :: bA : bQ$, it is as $Qq : Rr :: dC : dR$, and as $Rr : Ss :: fE : fS$, &c. And by compounding theſe proportions, it is as $Pp : Ss :: bA \times dC \times fE : bQ \times dR \times fS$. But by art. 237. it will be as $Pa : Ab :: Aa : bQ$, and by art. 367. as $PI : Pa :: dK : dR$. and by art. 370. as $PL : PI :: fM : fS$; and theſe three proportions being compounded, give $PL : Ab :: Aa \times dK \times fM : bQ \times dR \times fS$. But we had $Ss : Pp :: bQ \times dR \times fS : bA \times dC \times fE$; therefore $Ss \times PL : Pp \times Ab :: Aa \times dK \times fM : bA \times dC \times fE$. Whence $Ss = Pp \times \frac{Aa}{PL} \times \frac{dK}{dC} \times \frac{fM}{fE}$; or, by taking a line $V = Aa \times \frac{dK}{dC} \times \frac{fM}{fE}$, $Ss = Pp \times \frac{V}{PL}$.

375. *Corol.* 1. Hence the image Ss is equal to the object Pp, when PL is equal to V.

376. *Corol.* 2. The image Ss is as the object Pp directly and as PL inverſely.

377. *Corol.* 3. And therefore the image Ss is as the angle PLp which the object ſubtends at the principal focus L[b].

[a] Art. 245.

[b] Art. 222.

378. *Corol.* 4. The image is ſimilar to the object in all its parts. For when PL is given, Ss is as Pp.

379. *Corol.* 5. When the object is given, the image is inverſely as PL or directly as MS by art. 373.

380. *Corol.* 6. If the emergent rays be received upon a perpendicular plane cutting ηS in X and ηs in x, take a line $Ll = \frac{LI \times fM}{\eta M}$ and place it in a contrary direction to that of $M\eta$ from M, and the ſemidiameter Xx of this confuſed image will be equal to $\frac{Pp}{PL} \times \frac{\eta X}{\eta M} \times V$. For the triangles ηXx, ηSs being ſimilar, it is as $Xx : (Ss)\ \frac{Pp}{PL} \times V$ * $:: \eta X : \eta M + MS$, * Art. 374.
or $\eta M + \frac{LI \times fM}{PL}$* $:: \eta X \times PL : \eta M \times PL + LI \times fM :: \frac{\eta X \times PL}{\eta M} : PL$ * Art. 373.
$+ \frac{LI \times fM}{\eta M}$ or $PL + Ll$ or Pl by conſtruction. Whence $Xx = \frac{Pp}{Pl} \times \frac{\eta X}{\eta M} \times V$.

381. *Corol.* 7. Hence if the perpendicular plane be fixt at any place X, the confuſed image Xx is as $\frac{Pp}{Pl}$, or as the angle ſubtended by the object at the given point l*. * Art. 222.

Corol. 8. Hence if the object be given, this image Xx is inverſely as Pl; and is alſo ſimilar to the object.

PROPOSITION VI.

It is propoſed to find the ratio of the angles which the incident and emergent parts of a ray do make with each other and with the common axis of any number of given ſurfaces betwixt given mediums.

382. Things remaining as they were, let $PBDFS$ be the courſe of Fig. 392.
the ray; P and S its firſt and laſt interſections with the common axis of the ſurfaces. Take a line Z equal to $Ab \times \frac{dK}{cC} \times \frac{fM}{eE}$ &c. as far as the number of ſurfaces permits; and the angle APB will be to ESF as the given line Z to PL.

For $Qb : bA :: Aa : aP$ * and conjointly and alternately $Qb : Aa ::$* Art. 237.
$QA : AP ::$ ang. APB : ang. AQB*. In like manner, $Rd : Cc :: RC :$* Art. 222.
$CQ ::$ ang. CQD : ang CRD; and likewiſe $Sf : Ee :: SE : ER ::$ ang. ERF : ang. ESF; and ſo on. And by compounding theſe proportions it is as $Qb \times Rd \times Sf : Aa \times Cc \times Ee ::$ ang. P : ang. S. But in the demonſtration of the laſt propoſition we had $Aa \times dK \times fM : Qb \times Rd \times Sf :: PL : Ab$. Whence it eaſily follows that $Ab \times \frac{dK}{Cc} \times \frac{fM}{Ee} : PL ::$ ang. P : angle S. And by ſubſtracting the leſſer angle from the greater we have the

the angle made by the incident and emergent ray, as appears by producing theſe rays till they croſs one another.

383. *Corol.* 1. Let AB be the ſemiaperture of the firſt ſurface, and the ſemidiameter Mm of a perpendicular ſection of the emergent pencil of rays will be to AB, as the given line $\frac{LI \times fM}{Z}$ is to AP. For becauſe the ſubtenſes of ſmall angles are in a compound ratio of their legs and of the angles themſelves, we have Mm to AB in a ratio compounded of MS to AP, and of the angle MSm to the angle APB, or, by the propoſition, of PL to Z. Whence $Mm = \frac{AB}{AP} \times \frac{MS \times PL}{Z} = \frac{AB}{AP} \times \frac{LI \times fM}{Z}$*.

* Art. 371.

384. *Corol.* 2. And therefore Mm is as $\frac{AB}{AP}$, or as the angle APB.

385. *Corol.* 3. So that when AB is given, Mm is reciprocally as AP.

386. *Corol.* 4. If the emergent pencil be cut by a perpendicular plane at any other place X, the ſemidiameter Xx, of this ſection, is equal to $\frac{AB}{AP} \times \frac{LI \times fM - PL \times MX}{Z}$. For $Xx : (Mm)\ \frac{AB}{AP} \times \frac{LI \times fM}{Z} :: MS - MX : MS :: \frac{LI \times fM}{PL} - MX : \frac{LI \times fM}{PL} :: LI \times fM - PL \times MX : LI \times fM$.

Fig. 393.

387. *Corol.* 5. Hence we have the following conſtruction, take $L\Pi : LI :: fM : MX$ and place it in a contrary direction from L to that of MX from M; and in the line AG perpendicular to the axis, take AG to AB as MX to Z; then through G draw GH parallel to the axis, and with the aſymptotes GA, GH draw an hyperbola $\Pi\Upsilon$ through the point Π; and the perpendicular $P\Upsilon$ will every where equal the ſemidiameter Xx when the plane is fixt at X. For by the property of the hyperbola the rectangle $GH \times H\Upsilon = GA \times A\Pi$, whence $P\Upsilon = (H\Upsilon - GA =) \frac{GA \times A\Pi - GA \times AP}{AP} = \frac{AG}{AP} \times \overline{L\Pi - LP} = \frac{AG}{AP} \times \overline{\frac{LI \times fM}{MX} - LP} = \frac{AG}{AP} \times \frac{LI \times fM - LP \times MX}{MX} = \frac{AB}{AP} \times \frac{LI \times fM - LP \times MX}{Z}$ (by conſtruction) $= Xx$ in the 4th corol.

388. The line N or a middle proportional between LI and fM was made uſe of to determine the relation of any two conjugate focuſes P, S, to the principal focuſes L, M[a]; the line V or $Aa \times \frac{dK}{dC} \times \frac{fM}{fE}$, to determine the ratio of the object at P to its image at S[b]; and the line Z or $Ab \times \frac{dK}{cC} \times \frac{fM}{eE}$, to determine the ratio of the angles at P and S made by any ray with the axis of the ſurface[c]; I ſay the lines V, N, Z are continual proportionals in a ſubduplicate ratio of $Aa \times Cc \times Ee$ to $Ab \times Cd \times Ef$; and conſequently in lenſes they are equal to one another. For $V : Z :: \frac{Aa}{Cd \times Ef} : \frac{Ab}{Cc \times Ee} :: Aa \times Cc \times Ee : Ab \times Cd \times Ef$; and by art. 369 and

[a] Art. 373.

[b] Art. 374.

[c] Art. 382.

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366 it will appear that $N^2 : Z^2$ or $LI \times fM : \frac{Ab^2}{Cc^2 \times Ee^2} \times dK^2 \times fM^2 ::$ $(LI)\ \frac{Ia \times dK}{Ke} : \frac{Ab^2}{Cc^2 \times Ee^2} \times dK^2 \times \frac{eEf}{Ke} :: (Ia)\ \frac{aAb}{bc} : \frac{Ab^2}{Cc^2 \times Ee} \times \frac{cCd}{bc} \times Ef$ $:: \frac{Aa}{Cd \times Ef} : \frac{Ab}{Cc \times Ee} :: V : Z$, as before.

PROPOSITION VII.

To find the apparent diſtance of an object ſeen through any given ſyſtem of mediums; and to ſhew how it varies while the eye, object or ſyſtem is moved forward or backward.

389. Things remaining as they were, biſect LM in T, and in a perpendicular to the axis at T, take Tv, Tn, Tz ſeverally equal to the given lines V, N, Z; and through z, parallel to LM, draw lm cutting perpendiculars at L and M in l and m. Join mS and produce it both ways, and a perpendicular OX terminated at mS will be equal to the apparent diſtance of the object P ſeen from any point O. Fig. 394, 395.

In like manner let R be the conjugate focus of a pencil of rays ſuppoſed to flow from O; join lR and produce it, and the perpendicular PY terminated at lR will alſo be the apparent diſtance of the object P ſeen from O.

Laſtly take OG and PH each equal to TL or TM and place them inwards, if the order of the points LTM be according to the courſe of the rays that flow from P, otherwiſe outwards; and let lm produced cut the perpendiculars to the axis at G and H in g and h; then let gh be an ordinate to the axis of a parabola gZh, whoſe parameter belonging to the axis is the line Tv, and whoſe legs are extended from its vertex the ſame way as the perpendiculars Gg, Hh are extended from the axis of the ſyſtem; and the ordinate TZ will alſo be the apparent diſtance of the object P ſeen from O.

Now if the object and the ſyſtem be fixt while the eye is in motion along the axis of the ſyſtem, the moveable perpendicular OX, terminated at the fixt line mS, being always equal to the apparent diſtance, will ſhew how it varies. Or if the eye and the ſyſtem be fixt while the object is in motion along the axis of the ſyſtem, the moveable perpendicular PY, terminated at the fixt line lR, being always equal to the apparent diſtance, will ſhew how it varies. Or laſtly if the eye and object be fixt while the ſyſtem is in motion along its own axis, (ſuppoſing its parts to be kept at the ſame intervals,) the moveable perpendicular TZ terminated at the fixt parabola rZs, being always equal to the apparent diſtance, will ſhew how it varies.

And if one part of the ſyſtem be fixt, while the other is in motion, let the fixt image of the object formed by the fixt part, be at P; and by applying

ing the ſame conſtruction to the moveable part as before was applyed to the whole, the ordinate TZ will ſhew the apparent diſtance in this caſe.

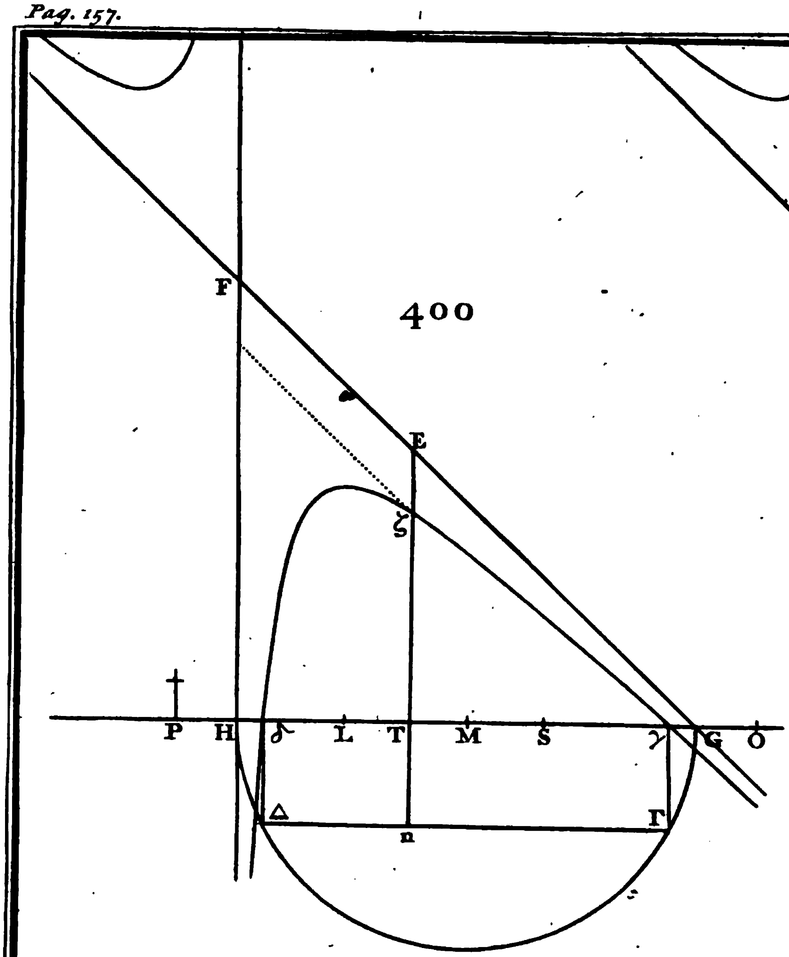
DEMONSTRATION.

Fig. 391. 390. For let the line $p\varpi$ drawn parallel to PO meet the viſual ray Os
produced, in ϖ, and compleat the rectangular parallelogram $Pp\varpi\Pi$;
a Art. 139. then $O\Pi$ is the apparent diſtance of the object Pp[a]; and the triangles
$O\Pi\varpi$, OSs being equiangular, we have $O\Pi : OS :: (\Pi\varpi$ or $Pp : Ss ::)$
b Art. 374. $PL : V$[b], that is ſuppoſing the perpendiculars OX, PY, TZ to be ſeve-
Fig. 394, 395. rally equal to the apparent diſtance $O\Pi$, we have OX or PY or $TZ : OS$
c Art. 373. $:: PL : Tv :: Tz : MS$; becauſe $PL \times MS = Tn^2 = Tv \times Tz$[c].
388. First then we have $OX : OS :: Tz$ or $Mm : MS$, which ſhews that mS
produced is the geometrick Place of the point X. Secondly we have alſo
$PY : Tz$ or $Ll :: (OS : MS ::) PR : LR$; for we had $PL \times MS = Tn^2$
$= OM \times LR$, whence $OM : MS :: PL : LR$ and disjointly $OS : MS ::$
$PR : LR$. And this ſhews that lR produced is the place of the point Y.
Laſtly we have $TZ = \left(\frac{PL \times OS}{Tv} = \frac{PL \times OM}{Tv} - \frac{PL \times MS}{Tv} = \right) \frac{HT \times TG}{Tv} - Tz$.
For by conſtruction $PL = HT$ and $OM = TG$ and $PL \times MS = Tv \times Tz$.
Now let the point T be removed to G; then becauſe $TG = 0$, by the e-
quation we have the ordinate $TZ = -Tz = Gg$. Again by removing T
to H, we have $TZ = -Tz = Hh$; and by the ſame equation we have
Fig. 396. $TZ + Tz$ that is $zZ = \frac{hz \times zg}{Tv}$. Biſect gh in A, and the point z being re-
moved to A, let zZ become $AB = \frac{hAg}{Tv} = \frac{hA^2}{Tv}$. Draw ZC perpendicular
* Euc. II. 5. to AB, then $BC = \left(AB - zZ = \frac{hA^2}{Tv} - \frac{hA^2 - Az^2}{Tv}{}^{*} = \frac{Az^2}{Tv} = \right) \frac{CZ^2}{Tv}$.
Which ſhews that the place of the point Z is a parabola whoſe parameter
belonging to the axis AB is Tv. Q. E. D.

Fig. 397. 391. While the ſyſtem remains fixt in any place, the apparent diſtance
of an object at P ſeen from O, will be to the apparent diſtance of an ob-
ject at O ſeen from P, as Z to V. For let a ray $PBDF$ fall upon an object
at O in any point K, draw $K\varkappa$ parallel to OP and let it meet the ray PB,
produced, in $\varkappa$, and compleat the rectangle $\varkappa KO\Omega$; then the apparent
* Art. 222. diſtance $P\Omega : PO ::$ angle OPK : angle $OP\varkappa$\*, or in a ratio compound-
ed of the angle OPK to OSK and of OSK to $\Omega P\varkappa$, that is of OS to OP
* Art. 382. and of PL to Z*. Therefore $P\Omega = \frac{PL \times OS}{Z}$. But we had $O\Pi = \frac{PL \times OS}{V}$
in the article above, and conſequently $O\Pi : P\Omega :: Z : V$.

Fig. 394, 395. 392. Hence the foregoing conſtruction will give the apparent diſtance
of an object at O ſeen from P, by drawing the line ln through v and by

making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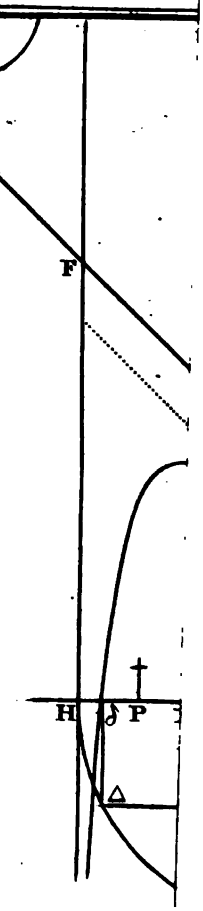

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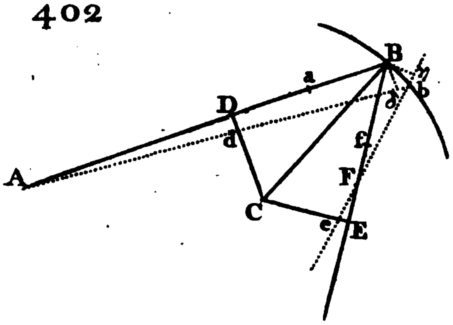

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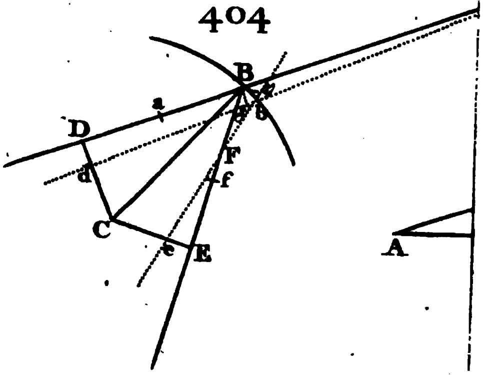

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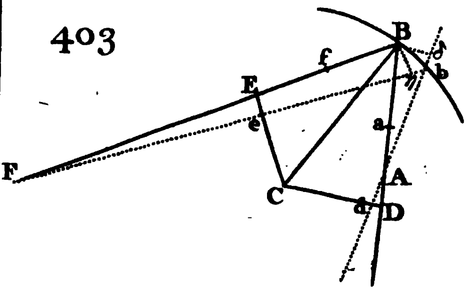

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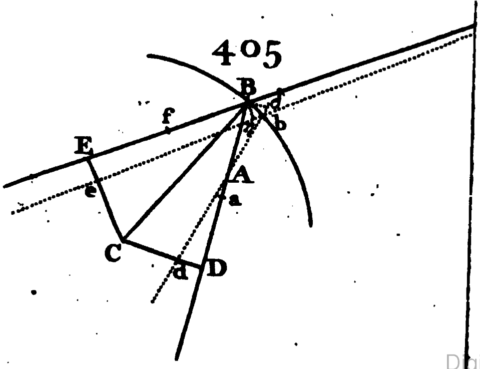

making Tz the parameter of the parabola. The ſame conſtruction ſerves alſo for a lens or lenſes, by drawing the line lm through n, and by making Tn the parameter of the parabola. For in lenſes the points n, v, z coincide [a]; and conſequently the apparent diſtances $O\Pi$ and $P\Omega$ are equal.

Fig. 398. 393. Hence if the places of the eye and object be given, and it be required to interpoſe a given ſyſtem of ſurfaces or lenſes in ſuch a place that the object ſhall appear through them all at a given diſtance; in any perpendicular DQD to the axis OPQ, on each ſide of Q take QD and QD equal to the given diſtance, and through the points D, D, parallel to the axis, draw the lines DZZ, DZZ cutting the parabola in four points Z, Z, Z, Z when poſſible; then draw the perpendiculars ZT &c. to the axis OP, and place that point of the given ſyſtem which biſects the interval LM at any of the four points T; and to the eye at O the object will appear at the given diſtance QD or TZ, as is evident by the foregoing conſtruction.

394. And by the ſame method, if the places of the object and ſyſtem be given, one may find the place of the eye, from whence the object ſhall appear at a given diſtance; and likewiſe if the places of the eye and ſyſtem be given, one may find the place of the object, where it ſhall appear at a given diſtance.

I have given the foregoing conſtructions of apparent diſtance and magnitude [b], for any one to examine how near they will anſwer his ideas of diſtance and magnitude upon making the experiments. And I add the following conſtructions for the diſtance of the laſt image of an object from the eye, to ſhew how widely it will differ from the apparent diſtance of the object, in moſt caſes.

PROPOSITION VIII.

To ſhew in what manner the diſtance between the eye and the laſt image of an object is varied, while the eye, object or refracting ſyſtem of any ſort, is moved forward or backward.

Fig. 399. 395. Things remaining as they were, take a perpendicular $M\sigma$ equal to MS, and draw σS cutting a perpendicular at O in x; and when the object and the ſyſtem are fixt, and the eye is in motion along OP, it is evident that Ox will be always equal to OS, the diſtance between the eye and the laſt image of the object P.

At L erect a perpendicular $L\omega$ equal to MO, and draw $\omega\psi$ parallel to LP, cutting a perpendicular at P in ψ; and through the focus R, with the center ω and aſymptotes ωL, $\omega\psi$ draw an hyperbola cutting the perpendicular at P in y; and Py will be the diſtance of the laſt image from the eye, while the eye and ſyſtem are fixt and the object is in motion along OP. For ſince $P\psi = L\omega = MO$, we ſhall have $Py = OS$ by taking $\psi y = (MS = \frac{Tn^2}{PL} =) \frac{Tn^2}{\omega\psi}$. Hence the rectangle $\omega\psi y = Tn^2$, which ſhews that the place of the point y is an hyperbola; which will paſs through R, becauſe the rectangle $RL\omega$ or $RL \times MO = Tn^2$.

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[a] Art. 388.

[b] Art. 140.

Fig. 400, 401. Laſtly erect a perpendicular $HF = HG$ and draw FG cutting the perpendicular nT produced in E, in which take $E\zeta = \frac{Tn^2}{HT}$ and place it downward if HT lyes from H toward G, otherwiſe upward, and through the point ζ, with the center F and aſymptotes FG, FH, deſcribe an hyperbola $\gamma\zeta\delta$; and while the eye and object are fixt, and the point T is in motion along with the ſyſtem, the ordinate $T\zeta$ will be always equal to the diſtance of the laſt image from the eye. For by conſtruction the line $E\zeta$ is reciprocally as HT, or reciprocally as FE, becauſe HT is to FE in a given ratio of HG to FG; that is, the magnitude of the rectangle under $FE\zeta$ is invariable, and conſequently the place of the point ζ is an hyperbola. Moreover by conſtruction $TE = TG = MO$, and therefore $T\zeta = (TE - E\zeta = MO - \frac{Tn^2}{HT} = MO - \frac{Tn^2}{PL} = MO - MS =) OS$.

396. Let the hyperbola $\gamma\zeta\delta$ cut the line OP in γ and δ; and when the point T in the given ſyſtem is brought to γ or δ, the laſt image of the object at P will fall upon the eye at O; becauſe $T\zeta = 0$.

397. Hence if two points O, P be given, and it be required to interpoſe a given lens or ſurface or a ſyſtem of lenſes or ſurfaces in ſuch a place, that the rays which belong to P ſhall belong to O after all the refractions, we have the ſolution in the foregoing article; or more readily in this manner. Parallel to PO, and through the point n, draw a line $\Gamma n \Delta$, cutting a ſemicircle upon the diameter GH in Γ, Δ; and the perpendiculars $\Gamma\gamma$, $\Delta\delta$ to the axis OP, will give the points γ, δ where T muſt be placed; for when OS or $T\zeta$ that is $TG - \frac{Tn^2}{TH} = 0$, we have the rectangle $HTG = Tn^2$
* Euc. VI. 13. $= \gamma\Gamma^2 = \delta\Delta^2 = H\gamma G = H\delta G$*; and conſequently T muſt coincide with γ or δ. But the *data* muſt be ſuch that Tn muſt be leſs than half HG; for if it be bigger, it will be impoſſible for the line $\Gamma n \Delta$ to cut the circle whoſe diameter is HG.

398. And when the diſtance of the image from the eye is diminiſhed to nothing, the apparent diſtance of the object becomes nothing at the ſame time. For the parabola in the foregoing propoſition will cut the line OP in the ſame points γ, δ as the hyperbola does, when poſſible, that is when Tn is leſs than half GH. For we had the parabolick ordinate TZ
* Art. 390. $= \frac{HT \times TG}{Tv} - Tz$*; which being put equal to nothing gives the rectangle $Tv \times Tz$ or $Tn^2 = HTG = H\gamma G$.

399. In all the propoſitions of this chapter I have only purſued the moſt general caſe, that is when the ſyſtem of ſurfaces or lenſes has two principal focuſes, as L and M; but the ſurfaces may be ſo ſituated that

the rays which fall parallel upon the firſt, ſhall emerge parallel from the laſt. Theſe particular caſes and ſome few determinations, which would have perplexed the reader too much and withdrawn his attention from the general caſe, I have purpoſely omitted. Whoever has a mind to conſider them by themſelves, will find them eaſy to be ſolved in the following manner. In the ſecond propoſition if the focuſes b, c be ſuppoſed to coincide, the principal focus I, K will be removed to an infinite diſtance. In this caſe Pa is to dR in the given ratio of the rectangle aAb to the rectangle cCd; and Pa, dR will lye the ſame way from a and d. For we had $Pa = \frac{aAb}{b\mathcal{Q}}$ and $dR = \frac{cCd}{c\mathcal{Q}}$; but $b\mathcal{Q} = c\mathcal{Q}$, becauſe $bc = 0$. In this and the like caſes when the ſyſtem has no principal focuſes, the geometrick places of the points X, Y, Z; x, y, ζ, in the foregoing conſtructions are all ſtraight lines.

Fig. 383 to 385.

400. By the analogy between the rules for finding the focus of a pencil of rays reflected and refracted at a ſingle ſpherical ſurface [a], it is manifeſt that all the propoſitions and conſtructions in this chapter, may be eaſily adapted to rays ſucceſſively reflected from any number of ſpherical ſurfaces betwixt any given mediums. And though I have mentioned none but ſpherical ſurfaces, yet all the concluſions are the ſame for any other ſurfaces of equal curvities to the ſpherical ones [b].

Deſign.

401. Likewiſe in the following chapter, where I conſider the reflections and refractions of rays that fall not only perpendicularly, or almoſt perpendicularly as before, but with any degree of obliquity upon any curves, I generally mention no other but a circle, which is always ſuppoſed to have the ſame curvity as any other curve, when poſſible, in the place where a ſlender pencil of rays is incident upon it. To find the radius of curvity, is the ſame thing as to draw perpendiculars to a given curve or to its tangents at the given points of incidence; and then to find the concourſe of two of the neareſt perpendiculars; for this point of concourſe is the center of a circle whoſe arch coincides with the given curve at the points of incidence. But this being a problem purely geometrical, it is ſufficient in this place to have given an idea of it; eſpecially ſince the conſideration of any other curves but circles, is ſeldom neceſſary for the ſolution of any optical appearances in nature. At firſt I ſhall conſider a ſuperficial pencil of rays, that lye all in one plane of incidence; becauſe it will appear afterwards that the rays of a ſolid pencil, which lye in different planes of incidence, will belong to different focuſes; and therefore inſtead of a ſpherical ſurface I ſubſtitute a great circle of that ſurface; conſidering it as a phyſical circle, not a mathematical one. Laſtly it is to be obſerved as before [c], that in ſtrictneſs of geometry the focus of a ſuperficial pencil of reflected or refracted rays, is nothing but the common interſection of two contiguous rays, whoſe points of incidence were in the middle of all the reſt; but the ſame interſection conſidered phyſically may be called the focus of a ſlender ſuperficial pencil of rays.

CHAP-

[a] Art. 207. 236.

[b] Art. 212.

[c] Art. 219.

CHAPTER IX.

Determinations of focuſes of rays falling with any degrees of obliquity upon any number of reflecting and refracting ſurfaces of any ſort, and alſo of the properties of Cauſticks.

PROPOSITION I.

Fig. 402 to 405.

402. LET *the ray* AB *belonging to the focus* A, *fall with any obliquity upon the concavity or convexity of a circle or any other curve, whoſe radius of curvity at* B *is* CB, *and be reflected along* BF *given in poſition; draw the ſines of incidence and reflection* CD, CE, *and biſect their equal coſines* BD, BE *in* a *and* f; *and ſay as* Aa *to* aB *ſo* Bf *to* fF, *and place* fF *the ſame way with reſpect to* Bf *as* Aa *lyes with reſpect to* aB; *and the point* F *will be the focus of a ſlender pencil reflected from an exceeding ſmall arch whoſe middle point is* B.

For let AbF be another ray reflected from b, the neareſt point to B; draw Cd and $B\delta$ perpendicular to Ab, and alſo Ce and $B\eta$ perpendicular to bF; and the line $B\delta$ will equal $B\eta$. For the angles $Bb\delta$, $Bb\eta$, which Ab and bF produced do make with the arch Bb or its tangent at b, are equal[a]; and conſequently the little right angled triangles $Bb\delta$, $Bb\eta$ are alſo equal. Again the lines Dd and Ee, which are the ſame at laſt, when b and B are coinciding, as the differences of the equal ſines CD and CE, Cd and Ce, are alſo equal. And ſince the triangles $BA\delta$, DAd are equiangular, and alſo $BF\eta$, EFe; the ratio of the diſtances BA to AD (or of $B\delta$ to Dd or of $B\eta$ to Ee) is the ſame as the ratio of the ſimilar diſtances BF to FE. Hence conjointly and disjointly $BA \pm AD : AD :: BF \pm FE : FE$, and alternately $\frac{BA+AD}{2} : \frac{BF+FE}{2} :: (AD : FE ::) \frac{BA-AD}{2} : \frac{BF-FE}{2}$, that is $Aa : Bf :: aB : fF$; and alternately $Aa : aB :: Bf : fF$. And ſuppoſing the incident rays to go backwards in theſe figures, they will be reflected from the convexity of the arch, in the ſame lines produced as before. *Q. E. D.*

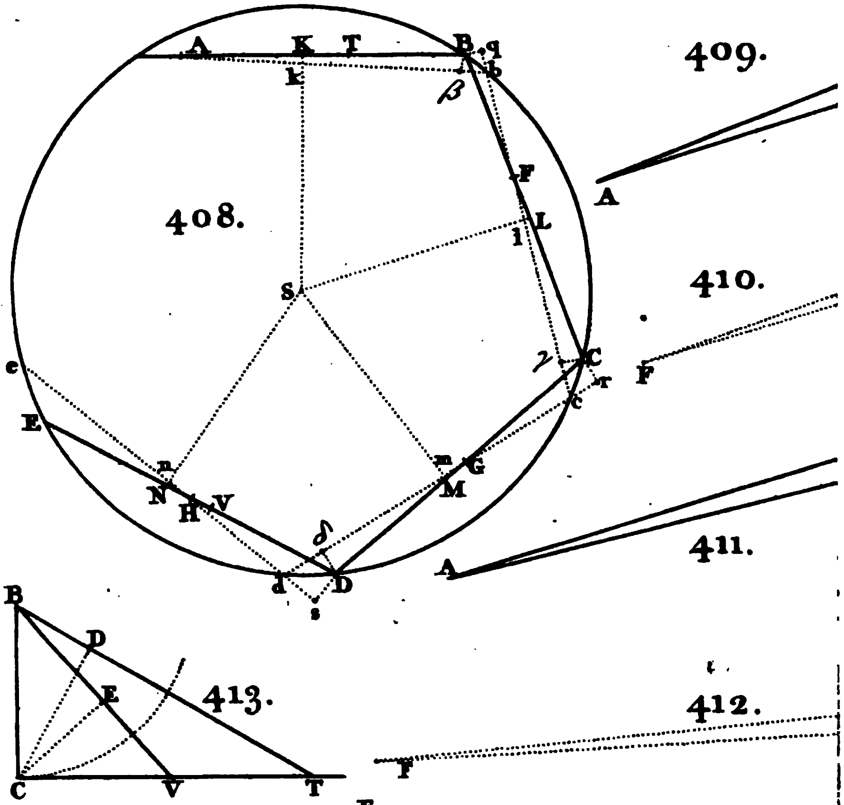
[a] Art. 9.

403. *Corol.* 1. When the incident ray AB paſſes through the center of the reflecting circle, this propoſition degenerates to the third propoſition, art. 207.

404. *Corol.* 2. It appeared in the demonſtration that the ratio of the diſtances BA to AD is the ſame as the ratio of the ſimilar diſtances BF to FE.

405. *Corol.* 3. The points a, f are the focuſes of ſlender pencils coming parallel to FB and AB reſpectively. For when the focus A or F is removed to an infinite diſtance the lines Dd, $B\delta$, $B\eta$, Ee become equal to one

another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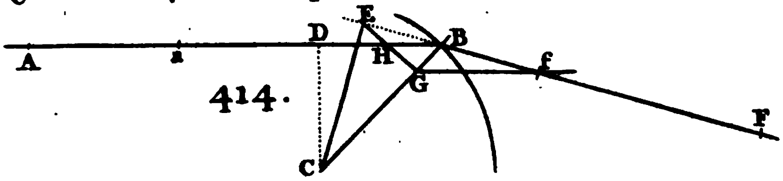
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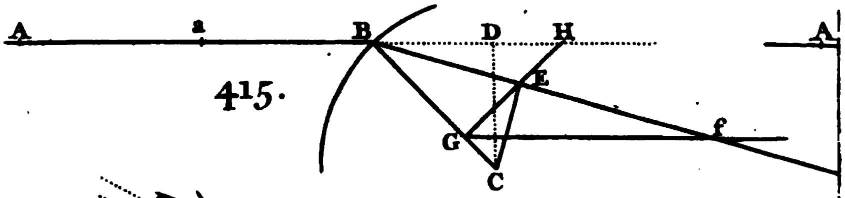
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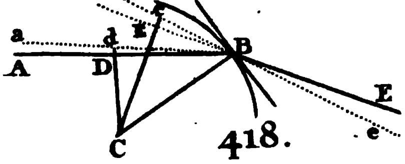
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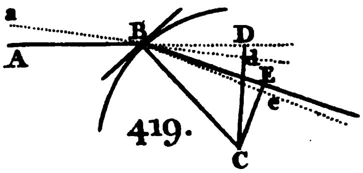
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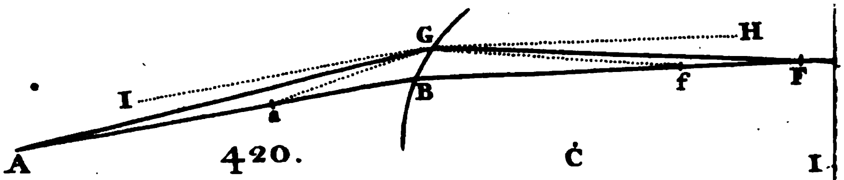

420.

another. Or it appears from the ſecond corollary, or from the propoſition it ſelf.

Fig. 406. 406. *Corol.* 4. Erect BG and BH perpendiculars to the incident and reflected rays AB and BF, and taking BH equal to BG, terminated at the axis AC produced, joyn CH and it will cut the reflected ray in the focus F. For drawing the equal ſines CD, CE, the triangles BAG, DAC and alſo BFH, EFC are equiangular. Whence $BA:AD::(BG:DC::BH:CE::)\ BF:EF$; which is the property of the focuſes A, F, by cor. 2.

Fig. 407. 407. *Corol.* 5. The focus A being given to find the point B, in a reflecting circle, from which and from the adjoining points, the rays ſhall be reflected into parallel lines when poſſible; in CA produced take AG equal to AC, and a circle deſcribed upon the diameter AG will cut the reflecting circle in the points B, B required. For drawing GB, the angle ABG in the ſemicircle is a right one, and conſequently in the equiangular triangles ABG, ADC, the ſides AB, AD are equal; and therefore A is the focus of rays that will be reflected from B into parallel lines by cor. 3.

408. The poſition of the reflected ray BF, which in this and the following propoſition is ſuppoſed to be given, may be determined by making the angle of incidence equal to the angle of reflection, or by inſcribing a chord in the reflecting circle, equal to the chord which the incident ray moved in; or by ſeveral other methods.

PROPOSITION II.

409. *The focus of incident rays being given, it is required to find their focus after any given number of ſucceſſive reflections from the inſide of a given circle.*

Fig. 408. Let $ABCDE$ be the given courſe of a ray reflected from a circle at B, C, D; and from the center S upon AB and DE, the firſt and laſt parts of the ray, draw the perpendiculars SK, SN; and from K and N towards B and D, the firſt and laſt points of reflection, take KT and NV ſeverally in proportion to KB or ND, as an unite to twice the number of ſucceſſive reflections; then let any point A in the line AB be the focus of incident rays, and ſay as TA to TK ſo VN to VH; and placing VH the ſame way with reſpect to VN as TA lyes with reſpect to TK, the point H will be their focus after all the reflections.

For let the perpendiculars SK, SL, SM, SN upon the ray AB, BC, CD, DE cut the next ray $Abcde$ in k, l, m, n; and let the focuſes or interſections of theſe two rays be at F, G, H. Alſo from B, C, D upon the ray Ab, bc, cd, draw the perpendiculars $B\beta$, $C\gamma$, $D\delta$, and likewiſe upon bc, cd, de draw the perpendiculars Bq, Cr, Ds. Theſe latter perpendiculars Bq, Cr, Ds are reſpectively equal to the former $B\beta$, $C\gamma$, $D\delta$; as it appeared in the demonſtration of the laſt propoſition. Now ſince FB,

FL, FC are in the ſame ratios to one another as Bq, Ll, $C\gamma$; and ſince $FB + 2FL = FC$, therefore $Bq + 2Ll = C\gamma$ or Cr. For the like reaſon $Cr + 2Mm = D\delta$ or Ds, and ſo on. Therefore ſince Kk, Ll, Mm are equal, as was ſhewn in the demonſtration of the laſt propoſition, the perpendiculars Bq, Cr, Ds, and their equals $B\beta$, $C\gamma$, $D\delta$, are ſeverally in arithmetical progreſſion; their common difference being $2Ll$ or $2Kk$. So that putting n for the number of chords BC, CD, that join the ſucceſſive points of reflection, made in the paſſage of the given ray from A to H, we have $B\beta + 2nKk = Ds$ the ſubtenſe of the angle DHS; that is, $Kk : B\beta + 2nKk :: (Kk$ or $Nn : Ds ::)$ $NH : HD$. Hence $2nKk : B\beta + 2nKk :: 2nNH : HD$, and disjointly $2nKk : B\beta :: 2nNH : HD - 2nNH$, and $Kk : B\beta$, that is $AK : AB :: NH : HD - 2nNH$, and disjointly $AK : KB :: NH : HD - 2nNH - NH$, or $ND - 2nNH - 2NH$. Therefore $AK : \frac{1}{2n+2} KB :: NH : \frac{1}{2n+2} ND - NH$, and conjointly $AK + \frac{1}{2n+2} KB : \frac{1}{2n+2} KB :: \frac{1}{2n+2} ND : \frac{1}{2n+2} ND - NH$; that is, taking $KT = \frac{1}{2n+2} KB$, and $NV = \frac{1}{2n+2} ND$, we have $TA : TK :: VN : VH$; and $2n+2$ or $2 \times \overline{n+1}$ is twice the number of reflections of the given ray AB; becauſe there is always one reflection more than the number of chords between the firſt and laſt points of reflection. *Q. E. D.*

410. *Corol.* 1. When AB is fixt in poſition VH is reciprocally as TA; and conſequently V is the focus of rays that come parallel to AB, and T the focus of rays that come parallel to ED: and theſe principal focuſes are found as above, by taking KT to KB as an unite to twice the number of ſucceſſive reflections.

411. *Corol.* 2. The perpendicular ſubtenſes Kk, Ll, Mm, &c. of the ſmall angles at A, F, G, &c. are equal to one another. And the ſmall arches Bb, Cc, Dd, &c. are in arithmetical progreſſion as well as the perpendiculars $B\beta$, $C\gamma$, $D\delta$, &c. becauſe the little right angled triangles $Bb\beta$, $Cc\gamma$, $Dd\delta$, &c. are equiangular.

PROPOSITION III.

Fig. 409 to 412.

412. *From the center* C *of a refracting circle* Bb, *betwixt given mediums, draw the perpendicular* CD *upon an incident ray* AB *and* CE *upon the refracted ray* BF *given in poſition; and if any point* A *be the focus of incident rays upon a ſmall arch* Bb, *and* F *be their focus after refraction, the ratio of the diſtances* BF *to* EF *will be compounded of the ratio of the ſimilar diſtances* BA *to* DA, *of the ratio of the ſines of incidence and refraction,* CD, CE, *taken directly, and of the ratio of the coſines,* BD, BE, *taken inverſely.*

For let AbF be the neareſt ray to ABF; and upon Ab draw the perpendiculars Cd, $B\delta$; and Ce, $B\eta$ upon bF; and the right angled triangles $B\delta b$, BDC will be ſimilar, as appears by taking the common angle DBb from the right angles CBb, $DB\delta$. And for the ſame reaſon the right angled triangles $B\eta b$, BEC will be ſimilar; and conſequently the whole figures $B\eta\delta b$, $BEDC$ will alſo be ſimilar. It is alſo to be obſerved that CD is to CE and Cd to Ce in the given ratio of the ſine of incidence to the ſine of refraction; and diſjointly Dd is to Ee in the ſame ratio. For the ſame line CDd may be conſidered as perpendicular to both rays AB, Ab when the angle BAb is vaniſhing[a]. Hence becauſe the triangles $BF\eta$, EFe are ſimilar; and alſo $BA\delta$, DAd; the ratio of BF to EF or of $B\eta$ to Ee, which is compounded of the ratios $B\eta$ to $B\delta$, $B\delta$ to Dd, Dd to Ee, is alſo compounded of theſe ratios reſpectively the ſame as the former, BE to BD, BA to DA, CD to CE. Q. E. D. The poſition of the refracted ray BF, which is ſuppoſed to be given in this and the following propoſitions, will be determined hereafter by lemma 4.

[a] Art. 204.

413. *Corol.* 1. When the incident rays are parallel, the ratio of BF to EF is compounded of the direct ratio of the ſines of incidence and refraction, and of the inverſe ratio of the coſines; becauſe the ratio of BA to DA is a ratio of equality when A is remote.

414. *Corol.* 2. The ratio of the perpendicular ſubtenſes Dd, Ee, of the ſmall angles at A and F, is invariable; being the ſame as of the ſine of incidence CD to the ſine of refraction CE.

LEMMA. I.

415. The ratio of the tangents, CT, CV, of any two angles, CBD, CBE, is compounded of the ratio of their ſines, CD, CE, taken directly, and of their coſines, BD, BE, taken inverſely. Fig. 413.

For the right angled triangles BCT, BDC are equiangular, and ſo are the right angled triangles BCV, BEC. Therefore the ratio of CT to CV, which is compounded of CT to CB and of CB to CV, or of CD to DB and of EB to EC, is the ſame as the ratio of the rectangle under CD, EB to the rectangle under DB, EC, which is compounded of the ratio of CD to CE and of EB to DB[b], that is of the ſines directly and coſines inverſely. Q. E. D.

[b] Euc. VI. 23.

PROPOSITION IV.

416. *Let* AB *and* Bf *be an incident and a refracted ray given in poſition. From* C *the center of curvity at* B, *or of the refracting circle, draw* CE *perpendicular to the refracted ray* Bf *produced; and let* Bf *be to* BE *as the tangent of the angle of incidence to the difference of the tangents of incidence and refraction; and let* Bf *be placed forward, that is according to the courſe* Fig. 414 to 417.

of the refracted ray, if the ſurface of the denſer medium be convex, otherwiſe backwards; and f *will be the focus of a ſlender pencil of rays that came parallel to* AB *upon the ſmalleſt arch at* B.

For the ratio of Bf to Ef, being compounded of the ratio of the ſines of the angles of incidence and refraction directly and of their coſines in- a Art. 413. verſely[a], is the ſame as the ratio of the tangents of thoſe angles[b]; and b Art. 415. disjointly Bf is to BE as the tangent of incidence to the difference of the tangents of incidence and refraction. The reaſon of the rule for the poſition of Bf is this, that the rays going forward in both caſes, will converge in one caſe and diverge in the other. *Q. E. D.*

417. *Corol.* 1. Hence Bf is to Ef as the tangent of incidence to the tangent of refraction.

418. *Corol.* 2. Let a be the focus of rays that came parallel to fB; then we have Bf to Da as BE to BD, that is, the continuations of the coſines BE, BD to the focuſes of rays that came parallel to them, are in the ſame ratio as the coſines themſelves. For Bf is to Ef as the tangent of incidence to the tangent of refraction, and Ba to Da in the ſame ratio inverted. Therefore $Bf : Ef :: Da : Ba$, and disjointly $Bf : BE :: Da : DB$, and alternately $Bf : Da :: BE : BD$.

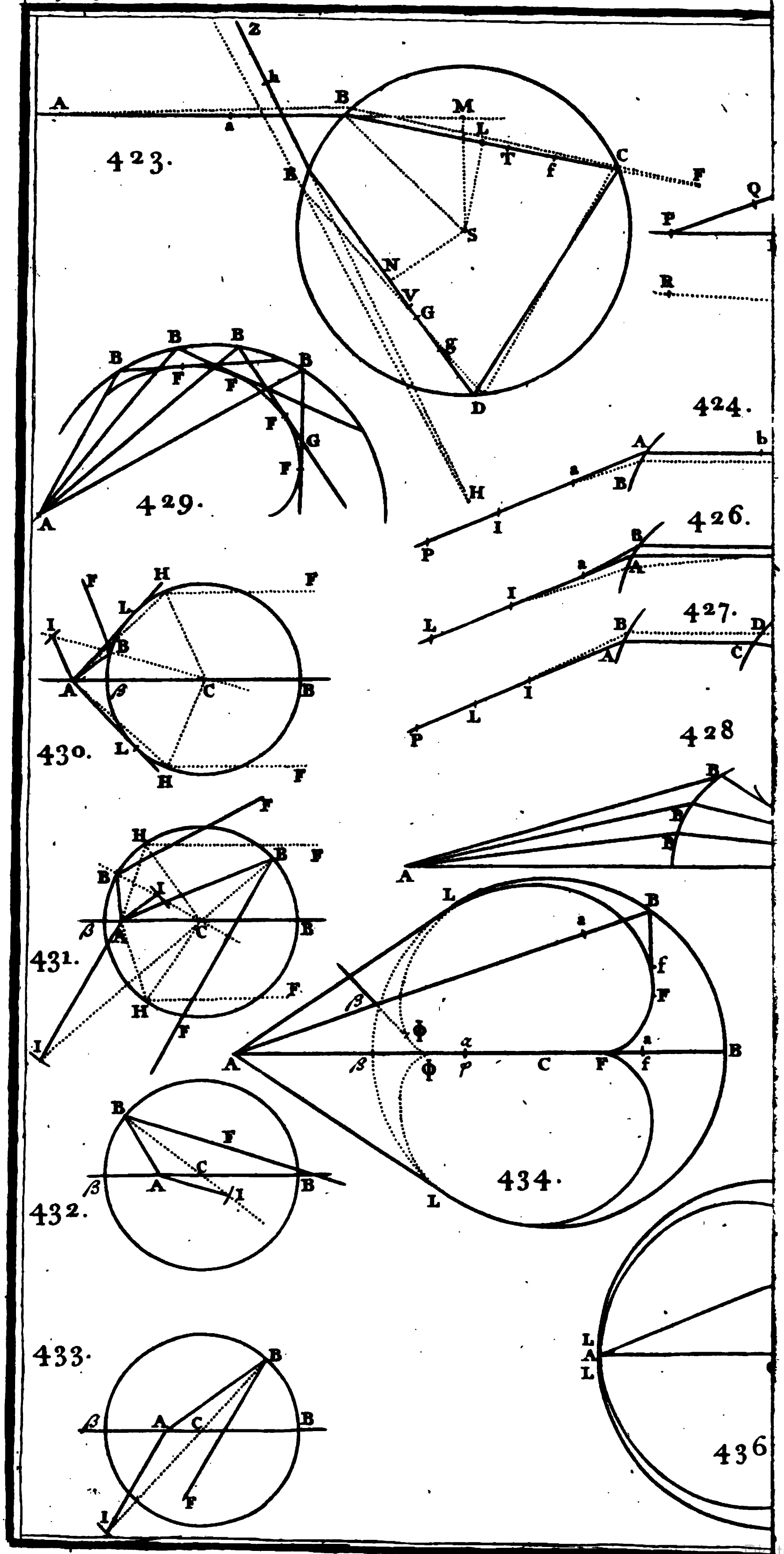
419. *Corol.* 3. The focus f may alſo be found by drawing lines in this manner. Draw GE perpendicular to the refracted ray Ef; EG perpendicular to the radius BC, and Gf parallel to the incident rays, and it will cut the refracted rays in their focus f. For let EG cut AB in H, and Gf being parallel to the baſe BH of the triangle BEH, we have $Bf : BE ::$ c Euc. VI. 2. $HG : HE$[c], the ſame proportion as in the propoſition: for GH and GE are tangents of the angles, GBH, GBE, of incidence and refraction.

420. *Corol.* 4. Hence it appears that while the angle of incidence continually increaſes, the focal diſtance Bf continually decreaſes, till it be reduced to nothing when the angle of refraction becomes a right one; or elſe till it be equal to the coſine of refraction when the angle of incidence is a right one. Conſequently the focal diſtance is the longeſt when the angle of incidence is the leaſt; and then this propoſition degenerates to d Art. 224. the ſecond propoſition of the third chapter[d]. For the tangents of very e Art. 204. ſmall angles are in the ſame ratio as their ſines or arches[e].

LEMMA II.

Fig. 418,419. 421. The leaſt increment of an angle of incidence, is to the contemporary increment of the angle of refraction, as the tangent of the angle of incidence, to the tangent of the angle of refraction.

Let two rays AB, aB, containing a very ſmall angle ABa, be refracted at B along the lines BE, Be by a plane or by any curve-ſurface. From any point C, of the line BC perpendicular to that ſurface, draw CDd f Art. 204. cutting the incident rays (produced) at right angles[f] in D and d; and like-


423.
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likewiſe CEe cutting the refracted rays (produced) at right angles in E and e. Then becauſe CD is to CE and Cd to Ce in the ſame ratio of the ſines, disjointly we have Dd to Ee as CD to CE. Now the ratio of the ſmall angles (ABa or) DBd and EBe, which are the contemporary increments or decrements of the angles of incidence and refraction, being compounded of the ratio Dd to Ee and of BE to BD[a], that is of CD to CE and of BE to BD, is the ſame as the ratio of the tangents of incidence and refraction[b]. *Q. E. D.*

[a] Art. 222.

[b] Art. 415.

422. *Corol.* Hence if the angles of incidence and refraction of either of the rays ABe, aBe be invariable, while thoſe of the other ray are varied a little, their ſmall increments or decrements will be always to each other in an invariable ratio.

PROPOSITION V.

Fig. 414 to 417.

423. *Let a ray* AB *falling with any obliquity upon a refracting curve at* B, *be refracted along* BF *given in poſition; and let* Ba *be the focal diſtance of rays coming parallel to* FB; *and* Bf *the focal diſtance of other rays coming parallel to* AB; *then ſuppoſing any point* A *to be a given focus of incident rays, ſay as* Aa *to* aB *ſo* Bf *to* fF, *and place* fF *the ſame way with reſpect to* fB *as* aA *lyes with reſpect to* aB; *and* F *will be the focus of the refacted rays.*

Fig. 420.

For let AGF be the neareſt ray to ABF; and joining aG and fG, a ray aG after refraction at G will go along a line GH parallel to BF, by the ſuppoſition; and likewiſe a ray fG will be refracted through G along GI parallel to BA. But the angle AGa is to the angle FGH or GFf in a certain given ratio[c]; and in like manner the angle aAG or AGI is to fGF in the ſame given ratio[c]. Therefore alternately the angle AGa : ang. aAG :: ang. GFf : ang. fGF; and the ſines of theſe ſmall angles are in the ſame proportion[d]; and conſequently in the triangles AGa, GFf, the ſides oppoſite to thoſe angles are alſo in the ſame proportion[e]; that is $Aa : aG :: Gf : fF$, or $Aa : aB :: Bf : fF$[f]. Let A approach towards a, and when it coincides with a, the line fF will become infinite; and therefore when A paſſes to the other ſide of a, the point F will alſo paſs from an infinite diſtance to the other ſide of f. *Q. E. D.*

[c] Art. 422.

[d] Art. 220.

[e] Art. 221.

[f] Art. 204.

424. *Corol.* 1. When the ray ABF is given in poſition the line fF is reciprocally as Aa.

Fig. 421.

425. *Corol.* 2. The ſame words applyed to the 421ſt figure are a demonſtration of the ſame proportion for the focus of reflected rays. For now the angle AGa is to FGH or GFf in a ratio of equality[g], and ſo is the angle aAG or AGI to the angle fGF; and conſequently the triangles AGa, GFf are equiangular; and therefore $Aa : aG$ or $aB :: Gf$ or $Bf : fF$. Which is another demonſtration of the firſt of theſe propoſitions[h]; and the points a, f which are the focuſes of parallel rays to FB

[g] Art. 8.

[h] Art. 402.

and

Fig. 422. and AB, may be found in this manner. Let two incident rays AB, ab
deſcribe two parallel chords of a circle $AabBcC$, and be reflected along
a Art. 9. 19. the chords BC, bc reſpectively equal to BA, ba[a], and let them croſs in f.
Then $2Bb = (Bb + Aa =$ arch $AB -$ arch $ab =$ arch $BC -$ arch $bc =) Cc$
$- Bb$. Hence $Cc = 3Bb$ and $Bb : Cc :: 1 : 3$. Let the chords Ab, ab approach
infinitely near to one another, and the figures Bfb, Cfc will become
b Art. 204. equiangular triangles[b], and conſequently $Bf : fc$ or $fC :: (Bb : Cc$
$::) 1 : 3$; therefore $Bf = \frac{1}{4} BC = \frac{1}{4} BA$.

PROPOSITION VI.

426. *Having the focus of a ſlender ſuperficial pencil of rays falling with any obliquity upon a great circle of a ſphere of any homogeneal matter, it is propoſed to find the focus of the rays that emerge from the ſphere by refraction, after any given number of ſucceſſive reflections within it.*

Fig. 423. Let $ABCDEZ$ be the courſe of a ray incident at B, ſucceſſively
c Art. 183 &c. reflected any given number of times, as at the points C, D[c], and emergent
at E. And let A be the given focus of incident rays; F their focus
after the firſt refraction at B; G their focus after the reflections at C and
D; and H their focus after emergence by refraction at E. To find theſe
focuſes, let Bf and Ba be the focal diſtances of rays coming parallel to
AB and FB reſpectively, and ſay as $Aa : aB :: Bf : fF$; which gives
the focus F, by placing fF the ſame way with reſpect to Bf as Aa lyes
* Art. 423. with reſpect to aB*. Again from S the center of the ſphere, draw SL,
SN perpendiculars to the firſt and laſt chords BC, DE, and towards C and
D the firſt and laſt points of reflection, take LT and NV ſeverally in proportion
to LC or ND, as an unite to twice the given number of ſucceſſive
reflections; then becauſe F is the focus of the rays incident at C, ſay as
$TF : TL :: TL$ or $VN : VG$, which gives the focus G after the laſt reflection
at D, by placing VG the ſame way with reſpect to VN as TF lyes
* Art. 409. with reſpect to TL*. Laſtly let Eg and Eh be the focal diſtances of rays
coming parallel to ZE and GE reſpectively, then becauſe G is the focus
of the rays incident at E, ſay as $Gg : gE :: Eh : hH$; which gives H the
focus of the emergent rays, by placing hH the ſame way with reſpect to
* Art. 423. Eh as Gg lyes with reſpect to gE*. *Q. E. J.*

427. *Corol.* 1. While the incident pencil revolves about the focus A
with an angular motion in the plane of the circle $BCDE$, the proportio-
d Art. 420. nable lines Aa, aB, Bf, fF will all vary their lengths[d], and ſo will TL;
and when TF, TL, Tf are continual proportionals the emergent rays will be parallel to EZ. For we had $Gg : gE :: Eh : hH$; therefore hH will be infinite when Gg is nothing, that is when VG equals Vg or Tf, which are always equal, by reaſon of the equal chords BC, DE and equal refractions at E and B. But by conſtruction TF, TL, VG are always continual

nual proportionals; therefore when VG and Tf are equal, we have TF, TL, Tf continual proportionals.

428. *Corol.* 2. Hence if the incident rays be parallel, the emergent rays will alſo be parallel when Tf and TL become equal to one another; and conſequently when Lf is to LC as an unite to the given number of ſucceſſive reflections. For when the focus A is infinitely remote, the focus F will coincide with f, and conſequently the continual proportionals TF, TL, Tf* will become equal to one another. * Art. 427.

429. *Corol.* 3. Hence drawing SM perpendicular to AB produced, and putting n for the given number of ſucceſſive reflections, if the incident rays be parallel, the emergent rays will alſo be parallel when $BL : BM :: \overline{n+1}, SL : SM$. For in Corol. 2, we had $n : 1 :: LC$ or $LB : Lf$, and conjointly $n+1$ to 1 (as Bf to Lf, that is,) in a ratio compounded of SM to SL and of BL to BM*; therefore by compounding * Art. 413. theſe ratios with the ratio of SL to SM, we have $\overline{n+1}, SL : SM :: BL : BM$.

430. *Corol.* 4. Hence putting $I : R :: SM : SL$, and $m = n+1$, we ſhall have $BM : BS :: \sqrt{II-RR} : \sqrt{\overline{mm-1}, RR}$; which determines the angle of incidence BSM when the rays that came parallel to AB ſhall emerge parallel to EZ. And this is Sir *Iſaac Newton*'s rule for deter- Opt. p. 148. mining the apparent diameters of the Rain-bows, as ſhall be explained hereafter. For becauſe $SM : SL :: I : R$, we have $SM \pm SL : SM :: I \pm R : I$. And in Corol. 3. we have $BL \pm BM : BM :: mR \pm I : I$. But $SM^2 + BM^2 = SL^2 + BL^2$*, or $SM^2 - SL^2 = BL^2 - BM^2$, * Euc. I. 47. or $\overline{SM+SL} \times \overline{SM-SL} = \overline{BL+BM} \times \overline{BL-BM}$. Whence $SM + SL : BL + BM :: BL - BM : SM - SL$; that is, by ſubſtituting the values of theſe terms, which are given by the foregoing proportions, $\frac{I+R}{I} SM : \frac{mR+I}{I} BM :: \frac{mR-I}{I} BM : \frac{I-R}{I} SM$. Hence $\overline{I+R} \times \overline{I-R} \times SM^2 = \overline{mR+I} \times \overline{mR-I} \times BM^2$, and conſequently $BM^2 : SM^2 :: II - RR : mmRR - II$; and conjointly $BM^2 : BS^2 :: II - RR : \overline{mm-1} \times RR$; and $BM : BS :: \sqrt{II-RR} : \sqrt{\overline{mm-1}, RR}$.

PROPOSITION VII.

431. *Having the poſitions of two given curves, betwixt three given mediums; and ſuppoſing the rays of a ſlender pencil in either of the outward mediums to be parallel, and to fall upon the curves with any obliquity; it is propoſed to find their focus after refractions through both the curves.*

Let $aAbcCd$ be a ray refracted at A and C by the given curves AB, Fig. 424. CD; in this given ray let a and d be the focuſes of rays, which before they emerged by refractions at AB and CD into the outward mediums, went both ways parallel and very near to AC in the inner medium. Alſo let b and c be the focuſes of other rays, which before they emerged by refractions into the inner medium, came parallel to aA and dC reſpectively in the outward mediums. Theſe focuſes may be found by art. 416 and 401. Say then as $cb : bA :: Aa : aI$, and placing aI the ſame way with reſpect to aA as bc lyes with reſpect to bA, the point I will be the focus of rays after both refractions, which came parallel from without upon * Art. 423. the curve CD*; becauſe c is their focus after the firſt refraction. In like manner by ſaying as $bc : cC :: Cd : dK$, and by placing dK the ſame way with reſpect to dC as cb lyes with reſpect to cC, the point K will be the focus of other rays after both refractions which came parallel from with- * Art. 423. out upon the other curve AB*. *Q. E. I.*

Fig. 425. 432. *Corol.* 1. Let the two ſurfaces AB, CD compoſe a ſphere of homogeneal matter, and let S be its center. Draw OM perpendicular to the incident ray aA produced and ON perpendicular to the emergent ray dC produced. Then biſect aM and dN in I and K, and the points I, K will be the focuſes of rays refracted through the ſphere that came parallel * Art. 418. to dC and aA. For biſecting AC in L, we have $Lc : LC :: Cd : CN$*, and conjointly $Lc : cC :: Cd : dN$; but by the propoſition we have bc or $2Lc : cC :: Cd : dK$, and conſequently $2Lc \times dK = (cC \times Cd =)\ Lc \times dN$ by the former proportion; therefore $dK = \frac{1}{2} dN$: and by the ſame argument $aI = \frac{1}{2} aM$.

433. *Corol.* 2. We had Cd to CN as the tangent of the leſſer of the an- a Art. 416. gles of incidence and refraction to the difference of their tangents[a]; and conſequently the focus K will lye without or within the ſphere, according as that leſſer tangent is bigger or leſs than the difference of the ſaid tangents; which difference increaſes to infinity, while the angles of incidence and refraction increaſe; that is while the ray AC goes farther and farther from the center of the ſphere.

PROPOSITION VIII.

434. *The focus of incident rays being given, it is propoſed to find their focus after both refractions through two given curves, betwixt three given mediums.*

Fig. 424. In the ray $IaACdK$ given in poſition, let a and d be the focuſes of rays, which before they emerged by refractions at AB and CD into the outward mediums, went both ways parallel and very near to AC in the inner medium. Alſo let I and K be the focuſes of other rays, which before their refractions through both the curves, came parallel to CK and AI in the outward mediums. Then let P be the focus of incident rays upon the ſurface AB; I and a the focuſes of thoſe rays which came the contrary way to the incident ones; and ſay as $PI : Ia :: dK : KR$, and

placing KR the ſame way with reſpect to Kd as IP lyes with reſpect to Ia, the point R will be their focus after both the refractions.

For by ſaying as $Pa : aA :: Ab : bQ$ and by placing bQ as uſual, the point Q will be their focus after the firſt refraction at AB*; the ſame point Q being the focus of incident rays upon CD, ſay again as $Qc : cC :: Cd : dR$, which being placed as uſual gives the focus R after both refractions[a]. Now by the firſt of theſe proportions and of thoſe in the foregoing propoſition, the rectangle $Pa \times bQ = (aA \times Ab =)\, bc \times aI$; and by the ſecond proportion in this and in the foregoing propoſition, the rectangle $Qc \times dR = (cC \times Cd =)\, bc \times dK$; then by reſolving the two former rectangles into the proportion of their ſides, it is as $Ia : Pa :: bQ : bc$, and disjointly (or conjointly) $PI : Pa :: (Qc : bc ::)\, dK : dR$ by reſolving the latter rectangles; and disjointly (or conjointly) $PI : Ia :: dK : KR$. Q. E. D.

* Art. 423.

[a] Art. 423.

435. *Corol.* 1. Hence when the ray $IACK$ is given in poſition, KR is reciprocally as PI; becauſe the rectangle $Ia \times dK$ is invariable.

436. *Corol.* 2. In the ſphere we have $PI : IM :: NK : KR$, which gives the focus R by placing KR the ſame way with reſpect to KN as IP lyes with reſpect to IM. Becauſe we had $IM = Ia$ and $KN = Kd$*. And this rule for the focus R is the very ſame as we had in the 236th article, when the rays paſs near the center of the ſphere.

Fig. 425.

* Art. 432.

437. *Corol.* 3. Draw PS cutting the circle in B and D; and let SF be the focal diſtance of the ſphere; then if SP be longer than SF, while the arch BA increaſes, the line KR will decreaſe to nothing, till all the points N, C, K, R coincide together in the ſurface of the ſphere. For while BA increaſes, AM decreaſes to nothing, being always half the chord of the arch cut off by PA produced. And IM decreaſes faſter than AM; being firſt bigger, then equal to and then leſs than AM*. But PI or $PA \mp AI$ increaſes; becauſe PA always increaſes, while AI firſt decreaſes to nothing, and then increaſes on the oppoſite ſide to AP. Therefore KR or $\frac{IM \times NK}{PI}$ decreaſes to nothing.

* Art. 433.

438. *Corol.* 4. If SP be leſs than the focal diſtance SF, while the arch BA increaſes from nothing, the negative line KR will firſt increaſe to infinity and then decreaſe to nothing as before. For when A coincides with B, the points M, I coincide with S, F; and while BA increaſes, MI decreaſes[b] till PI becomes nothing, and KR infinite[c]; after this KR decreaſes to nothing as before.

[b] Art. 437.

[c] Art. 435.

439. *Corol.* 5. Hence it appears that NR firſt increaſes to infinity and then decreaſes to nothing.

440. *Corol.* 6. While NR is decreaſing the point R will enter the ſphere and return back again to its ſurface before NR vaniſhes at the ſurface. In the line PI take $QI : IM :: NK : KC$, and while AB increaſes, theſe

two ratios will at laſt become ratios of equality; (becauſe the laſt ratio of NC to Cd will be infinite[a];) therefore QI will be leſs than PI. But $PI:IM::NK:KR$*; and conſequently $PI\times KR=QI\times KC$, and $PI:QI::KC:KR$, and therefore KR is leſs than KC; and ſo the focus R will enter the ſphere; and will return again to its ſurface, becauſe NR vaniſhes there, when the incident and emergent rays are tangents to the ſphere.

a Art. 433.
* Art. 436.

PROPOSITION IX.

441. *Having the poſition of three given curves, betwixt four given mediums, and ſuppoſing the rays of a ſlender pencil in either of the outward mediums to be parallel, and to fall with any obliquity upon the curves, it is propoſed to find their focus after all the refractions.*

Fig. 426. Suppoſing the focuſes a, d and I, K to be determined by the 8th propoſition for two contiguous ſurfaces, as AB, CD; let the ray CeE be refracted at the ſurface EF along EfM; and let rays coming parallel to ME and CE have their focuſes at e and f after refraction at EF only; whereof e being the focus of incident rays upon the ſurface CD, ſay as $eK:Kd::aI:IL$, and placing IL as uſual, the point L is the focus of the rays after three refractions that came parallel from without upon the curve EF*. Again ſay as $Ke:eE::Ef:fM$, and placing fM as uſual, the point M is the focus of other rays after three refractions, which came parallel from without upon the curve AB*. Q. E. I.

* Art. 434.
* Art. 423.

PROPOSITION X.

442. *Having the focus of a ſlender pencil of rays, incident with any obliquity upon any given number of curves betwixt given mediums, it is propoſed to find the focus of the emergent rays.*

Fig. 427. Let I and f be the focuſes of rays, which before they emerged by refractions into the two outward mediums, went both ways parallel and very near to the given ray in either of the inner mediums, ſuppoſe CE; and let L and M be the focuſes of other rays, which before refractions through all the curves, came parallel to ME and LA reſpectively. Then let P be the given focus of the incident rays, L and I the focuſes of thoſe other rays which came the contrary way to the incident ones; and by ſaying as $PL:LI::fM:MS$, and placing MS the ſame way with reſpect to Mf as LP lyes with reſpect to LI, the point S will be their focus after all the refractions.

This may be demonſtrated by the 9th propoſition in the very ſame manner as the 8th was demonſtrated from the 7th, or in the very words of the 370th and 371ſt articles. And by the method of theſe demonſtrations it appears that the rule here delivered for three ſurfaces ſerves univerſally for any other number. But to avoid being tedious I have omitted ſome caſes of theſe propoſitions, as I did of thoſe in the 8th chapter, ſuch as are mentioned in the 399th article.

443. *Corol.* 1. Hence when the ray *PLACEMS* is given in poſition, *PL* is reciprocally as *MS*, becauſe the rectangle under them is equal to the invariable rectangle under *LI*, *fM*.

444. *Corol.* 2. The ſame rules and demonſtrations will ſerve alſo for finding the focus of rays ſucceſſively reflected from any given number of curves, by quoting the firſt of theſe propoſitions inſtead of the fifth.

DEFINITION OF A CAUSTICK

445. When an infinite number of incident rays, *AB*, *AB*, &c. that lye all in one plane of incidence, ſhall not belong to a ſingle point or focus after their laſt reflections or refractions, but ſhall cut one another in an infinite number of points; if a curve *FFF* be ſuppoſed of ſuch a ſhape as to touch every one of the reflected or refracted rays *BF*, *BF*, &c. produced if need be, in the points *F*, *F*, &c. the curve *FFF* is called *a Cauſtick by reflection or by refraction*, according as the name is applied to reflected or refracted rays. Fig. 428, 429.

446. *Corol.* 1. Let any two tangents *BF*, *BF* interſect one another in *G*; and ſuppoſing theſe tangents to approach to one another till they coincide, the points of contact and the point of interſection will alſo approach to one another till they alſo coincide. It is manifeſt therefore that a reflected or refracted ray touches the cauſtick in that point of the ray, where its interſection with the next ray vaniſhed, when they were ſuppoſed to coincide.

447. *Corol.* 2. Therefore conceiving two incident rays infinitely near to each other to revolve about their focus *A*, in the plane of incidence, the focus *F* or interſection of the reflected or refracted rays will deſcribe the cauſtick above defined. Which is called a real or an imaginary cauſtick according as *F* is the focus of converging or diverging rays. To get an idea of the ſhapes of cauſticks formed by rays reflected from a circle or any curve, it is requiſite to conſider the different inclinations of the reflected rays to that diameter of the circle which paſſes through the focus of incident rays, or to the axis of the curve.

LEMMA III.

448. Let *A* be the focus of incident rays upon a circle *βBH* whoſe center is *C*. In the angle of incidence *ABC*, or in its complement to two right angles, inſcribe a line *AI* equal to the incident ray *AB*; and the reflected ray *BF* will be parallel to *AI*. Fig. 430 to 433.

For the angles *ABI*, *AIB*, *IBF* are equal to one another[a]. *Q. E. D.*

449. *Corol.* 1. Draw the axis *AC*, cutting the circle at right angles in *β* and *B*; and let the lines *AL*, *AL* touch it in *L* and *L*; and the points

[a] Euc. I. 5, 29

L, L will divide the circumference into two arches whereof LBL, the remoter from A, will be concave towards A, and $L\beta L$ the nearer to A will be convex towards it. And when the point B comes to L the point I will alſo coincide with it, and ſo the reflected ray will proceed in the direction of the incident ray.

450. *Corol.* 2. In the reflecting circle inſcribe two lines AH, AH ſeverally equal to AC, the diſtance of the focus from the center; and the rays that are reflected from the remoter arch HBH will all converge towards the axis AC; and thoſe which are reflected from the nearer arch $H\beta H$ will all diverge from AC; and the two rays, HF, HF, that are reflected from the points H, H, being parallel to the axis AC, will divide the converging rays from the diverging ones. For ſuppoſing the ray AB to approach to AH and to coincide with it, the line AI will approach to AC and coincide with it; and then the reflected ray BF, which is always parallel to AI, will now become parallel to AC. But while AB is ſhorter than AH or AC, the equal line AI will alſo be ſhorter than AC, and conſequently will be ſituated on the ſame ſide of the axis AC as AB, and therefore the reflected ray BF, being parallel to AI, will diverge from the axis AC. On the other hand while AB is longer than AH or AC, the line AI will alſo be longer than AC, and conſequently will be ſituated on the contrary ſide of the axis, AC, to AB; and therefore BF being parallel to AI, will now converge toward the axis AC.

451. *Corol*, 3. Hence it is evident, that when the focus A is nearer to the center of the circle than half a ſemidiameter, the rays reflected from the whole circle will all converge towards the axis AC.

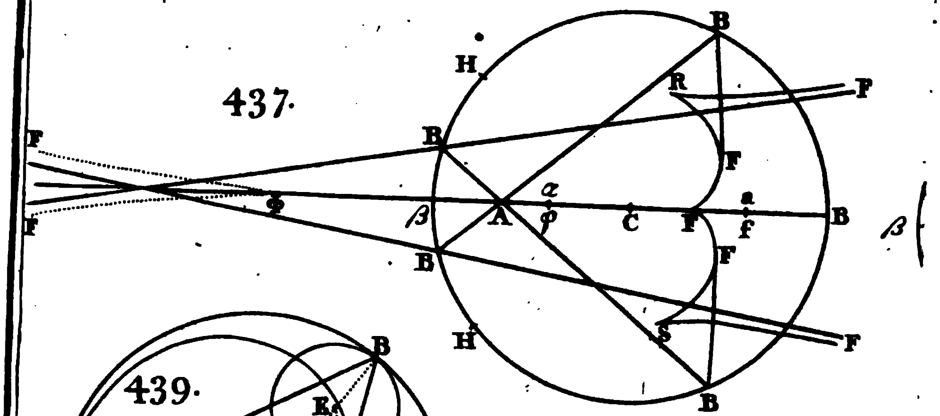
PROPOSITION XI.

Concerning the ſhapes and properties of cauſticks by reflection.

Fig. 434. 452. *Caſe* 1. Let A be the focus of incident rays, and be out of the reflecting circle $LBL\beta$; and let the lines AL, AL, touch it in L and L; and while the point of incidence B of the ray AB is deſcribing the arch $LBBL$ which is concave towards the focus A, the conjugate focus F of an infinitely ſlender pencil, will deſcribe a real cauſtick $LFFL$*; and while the point of incidence β is deſcribing the arch $L\beta\beta L$ which is convex towards the focus A, the conjugate focus Φ will deſcribe an imaginary cauſtick $L\Phi\Phi L$*. The real and imaginary cauſticks are both contained within the reflecting circle; and each of them conſiſts of two ſimilar and equal parts on oppoſite ſides of the axis AC, where each couple of ſimilar parts unite, and form two cuſps or ſharp points at F and Φ on oppoſite ſides of the center C. All this appears by the 402d article, by which $Bf=Ba=\frac{1}{4}B\beta$ and $Aa:aB::Bf:fF$, and therefore the cauſtick's tangent BF increaſes continually while AB and $A\beta$ move from AL towards the axis AC.

* Art. 447.

* Art. 447.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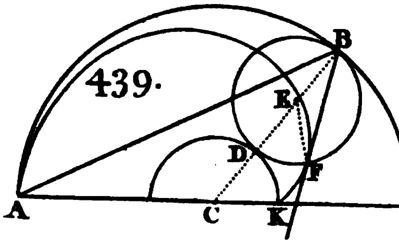
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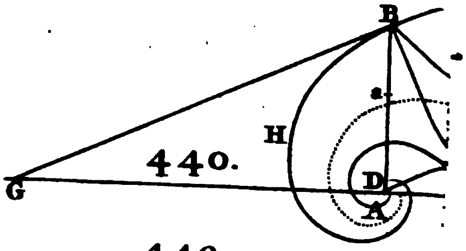
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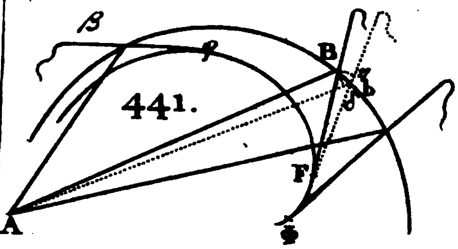
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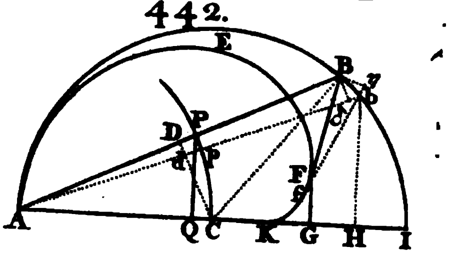
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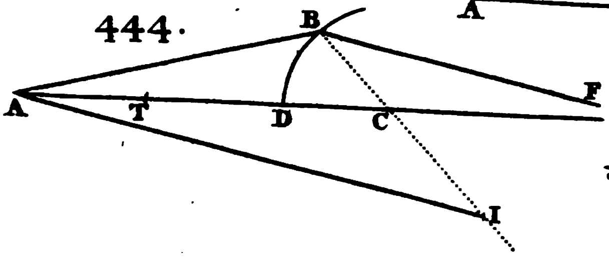
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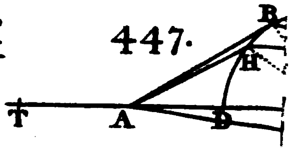
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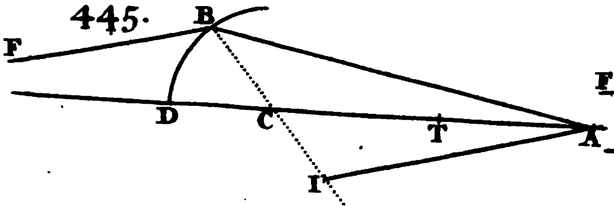
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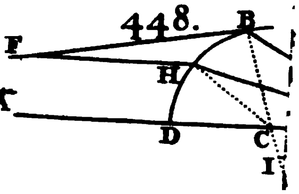
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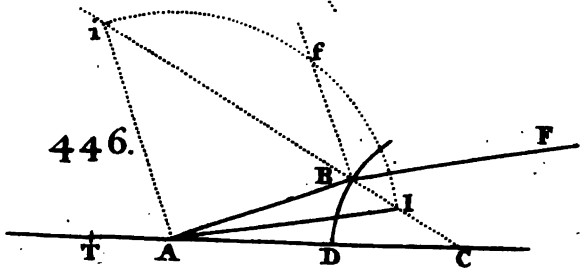
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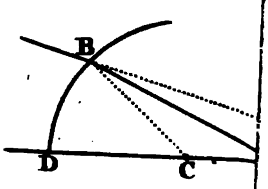


453. *Caſe* 2. Let the focus *A* be removed to an infinite diſtance; and the cauſtick's tangent *BF* will every where be equal to *Bf* or *Ba* or one quarter of the incident chord *Bβ*, ſuppoſed to move parallel to the axis *AC*. Fig. 435.

454. *Caſe* 3. Let the focus *A* be in the circumference of the reflecting circle, and the cauſtick's tangent *BF* will be every where equal to one third of the reflected or incident chord *AB**. For *Bf* or *Ba* being equal to $\frac{1}{4}$ *AB*, we have *Aa* to *aB* as *Bf* to *fF* as 3 to 1; and therefore $fF = \frac{1}{12}AB$ which being added to *Bf* or $\frac{3}{12}AB$ gives $BF = \frac{1}{3}AB$. The imaginary cauſtick is here vaniſhed and the parts of the real one come quite round to the focus *A* and there touch the circle. Fig. 436.

* Art. 402.

455. *Caſe* 4. Let the focus *A* enter the reflecting circle, but not ſo far as $\frac{1}{4}$ of its diameter *βB*; and the rays reflected from the neareſt point *β* will diverge from an imaginary cuſp Φ in this diameter produced backwards[a]. Beſides this cuſp at Φ and the correſpondent cuſp at *F*, formed by reflections from the other end *B* of the ſaid diameter, there are two other cuſps *R* and *S* on oppoſite ſides of this diameter. Theſe three cuſps belong to the real cauſtick, which has alſo two legs *RF*, *SF* infinitely extended, that approach to two aſymptotes *BF*, *BF*; to which there alſo belong two other legs Φ*F*, Φ*F* of an imaginary cauſtick approaching to their contrary ſides, and extended contrary ways to the former. For let the point of incidence *B* revolve round the circle beginning from the neareſt point *β*; and ſo long as the incident ray *AB* continues leſs than $\frac{1}{4}$ of the incident chord *BAB*, the reflected rays will diverge from the imaginary cauſtick Φ*F**; and when *AB* equals $\frac{1}{4}$ of its chord, the reflected ray becomes an aſymptote *BF*, or a tangent to the curve at an infinite diſtance[b]; and conſequently when *AB* is bigger than $\frac{1}{4}$ of its chord, the reflected rays will converge towards one another and form an oppoſite leg *RF* by the motion of their focus *F*; which firſt approaches towards the point of incidence *B* till it comes to a certain limit, and then recedes from it, as the incident chord grows ſtill longer; ſo that by means of theſe contrary motions of *F* in the reflected ray *BF*, and of the gradual change of its inclination, a cuſp or point of retrogreſſion will be formed at *R*; after the manner that the cuſp or point of retrogreſſion is formed in the axis at *F* by the lengthening and ſhortening of *BF*, and by the continual change of its inclination. Fig. 437.

[a] Art. 207.

* Art. 402.

[b] Art. 405.

456. *Caſe* 5. Let the focus *A* biſect the ſemidiameter *βC*, and the cuſp Φ being now removed to an infinite diſtance[c], the aſymptotes *BF*, *BF* will coincide with the axis; and when *A* advances nearer to the center, the two legs *RF*, *SF* will meet at a finite diſtance *A*Φ on the contrary ſide of *A*; and ſo this cauſtick will have 4 real cuſps; and when *A* comes into the center it will be contracted into this ſingle point. Fig. 437, 438.

[c] Art. 405.

457.

457. *Caſe* 6. Laſtly if the incident rays fall upon the oppoſite ſide of the circle converging towards *A*, all theſe cauſticks will be the very ſame as before, except that the real and imaginary parts will change places.

Fig. 439. 458. Some of theſe cauſticks may be deſcribed after the manner of deſcribing a cycloid. For example, in the third caſe where the focus *A* was in the circumference at *A*, and *BF* the tangent of the cauſtick was $\frac{1}{3}$ of the chord *AB*, of the reflecting circle; divide any ſemidiameter *BC* into 3 equal parts *CD*, *DE*, *EB*, and with the center *C* and ſemidiameter *CD* deſcribe the circle *DK*, cutting *AC* produced in *K*; alſo with the center *E* and ſemidiameter *ED* or *EB* deſcribe the circle *BFD*, cutting the reflected ray *BF* in *F*; then if the latter circle *BFD* be rolled, like a wheel, without ſliding upon the convexity of the former circle *DK*, the given point *F*, of the moveable circle, will deſcribe the cauſtick *AFK*. For joining *EF*, ſince the equicrural triangles *BEF*, *BCA* are equiangular, and ſince *BE* is $\frac{1}{3}$ *BC*, the line *BF* is therefore $\frac{1}{3}$ of *BA*; and conſequent-
a Art. 454. ly the point *F* is in the cauſtick[a]. And ſince the angles *DEF*, *DCK*, which are the complements of the equal angles *BEF*, *BCA*, are equal, the arches *DF*, *DK*, deſcribed with equal ſemidiameters are alſo equal.

459. The cauſtick of any curve, whoſe radius of curvity at every
Fig. 440. point is known, may alſo be determined by art. 402. For inſtance, let *AHB* be an equiangular ſpiral ſurrounding its pole *A*; whoſe eſſential property is this, that while the revolving ray *AB* increaſes or decreaſes, the magnitude of the angle *ABG*, made by the ray *AG* and the curve at *B* or its tangent *BG*, is invariable. Let the line *GAC* be always at right angles to *AB* and let it meet *BC*, perpendicular to *BG* in *C*; and in like manner as the point *B* of the moveable angle *ABG* deſcribes the ſpiral *BH*, ſo the point *C* of the equal angle *ACB* will deſcribe another equiangular ſpiral *ACI* about the pole *A*. From hence it appears that *C* is the center and *CB* the ſemidiameter of a circle of equal curvity to the firſt ſpiral at *B*, becauſe it is perpendicular to it at *B* and a tangent to the other ſpiral at *C*.

Now ſuppoſing *A* the focus of incident rays upon the ſpiral *AHB*, the focus *F* of the reflected rays, as *BF*, will deſcribe a third equiangular ſpiral *AF*, differing from either of the former in poſition only. For, from the center *C* of a circle of equal curvity to the ſpiral at *B*, let down the ſines of incidence and reflection *CD*, *CE* upon *AB*, *BF*; and ſince *D* always
* Art. 402. falls upon the focus *A*, the conjugate focus *F* will always fall upon *E**. And becauſe *AB*, *BE* are equal, and equally inclined to the perpendicular *BC* and conſequently to the tangent *BG*, the line *AF* will be parallel to *BG*; and ſo the angle *AFB*, made by *AF* and *BF*, the tangent of the cauſtick at *F*, will be equal to the invariable angle *ABG*. Q. E. D.

Fig. 441. 460. The length of any portion of a cauſtick formed by any reflecting curve, is equal to the ſum of the incident and of the reflected ray which terminates one extremity of that portion, diminiſhed by the ſum of the incident and of the reflected ray which terminates its other extremity.

Imagine the tangent BF to be a flexible line or ſtring, which being extended both ways does lap or unlap it ſelf upon the convexity of the cauſtick without ſliding, ſo as to meaſure the length of any portion of it to which it is applied. And having made the ſame conſtruction as in prop. 1, ſince the triangles $Bb\delta$, $Bb\eta$ were there ſhewn to be equal, the increment $b\delta$ of the incident ray BA, is every where equal to the decrement $b\eta$ of the ſtring $BF\Phi$, reckoning from any fixt point Φ. And if the point B moves the contrary way, the decrements of AB are every where equal to the increments of the ſtring. Taking therefore the correſpondent ſums of theſe increments or decrements, it follows, when AB, BF are in any other place as $A\beta$, $\beta\Phi$, that when AB increaſes, $A\beta - AB = \Phi F + FB - \Phi\beta$; whence by taking equal things from equal things, $A\beta + \beta\Phi - AB - BF = F\Phi$, the portion of the cauſtick: and when AB decreaſes, $AB - A\beta = F\varphi + \varphi\beta - FB$; whence $AB + BF - A\beta - \beta\Phi = F\varphi$.

461. In the third caſe when A is in the circumference, it follows from Fig. 436.
the rule above that the length of the portion $AF = AB + BF = \frac{4}{3}AB$: Fig. 435.
and in the 2d caſe, when the incident rays are parallel, the portion $LF = DB + BF = \frac{3}{2}DB$, the line DB being half $B\beta$.

462. The denſity of the rays in any particle of a cauſtick may be deter- Fig. 442.
mined in this manner. Let the incident rays AB, Ab be reflected from a ſmall arch Bb of any curve BI whoſe axis is AI; and let the reflected rays touch the cauſtick FfK in F and f. With the center A and with any given ſemidiameter AC deſcribe an arch CpP cutting the incident rays AB, Ab in P and p, and the denſity of the rays in the ſmall arch Ff, will be to the uniform denſity of the ſame rays in the arch Pp as Pp to Ff. For ſuppoſing all the rays incident upon the arch Bb to be regularly reflected, the ſame number of rays will exiſt every moment in the lines Pp, Ff; and conſequently their denſities in theſe lines are reciprocally as their lengths. Hence if the magnitude of the arch Pp be ſuppoſed invariable, the denſity of the rays in the particle Ff will be reciprocally as its length.

463. Draw the lines PQ, FG perpendicular to the axis AI, and when the whole figure is turned about this axis, all the rays that flow from A and are reflected from the ſurface deſcribed by the curve BI, will touch a ſuperficial cauſtick deſcribed by the circular motion of the linear cauſtick $EFfK$; and the denſity of the rays in any part of this cauſtick deſcribed by a ſmall arch Ff, will be to the uniform denſity of the incident rays in the ſpherical ſurface deſcribed by the arch CPp, as the rectangle under Pp and PQ to the rectangle under Ff and FG. For the ſame number of rays exiſts every moment in the rings deſcribed by the circular motion of the lines Pp, Ff; and their denſities, being uniform in each ring, are reciprocally as the magnitudes of the rings. But the ring deſcribed by

Pp is equal to the rectangle under Pp and the periphery deſcribed by the point P, and the ring deſcribed by Ff is equal to the rectangle under Ff and the periphery deſcribed by the point F; and ſince the ratio of theſe peripheries is the ſame as the ratio of their ſemidiameters, the former rectangle is to the latter as $Pp \times PQ$ to $Ff \times FG$.

464. To give an inſtance or two of this latter rule, let the reflecting ſurface ABI be ſpherical and let the focus A be in this ſurface whoſe center is C; and let the ſemidiameter AP equal AC. And the denſity of the rays in the ſuperficial cauſtick at F, will be to the uniform denſity of the incident rays in the ſpherical ſurface CP, as the ſemidiameter AC to $\frac{8}{3}$ of the ordinate FG. For the length of the portion AEF, of the linear cauſtick, is equal to $\frac{4}{3}$ AB* and conſequently the leaſt increment or decrement of this cauſtick is equal to $\frac{4}{3}$ of the increment or decrement of AB, that is $Ff = \frac{4}{3} b\delta$. Draw CD perpendicular to AP and we have CD equal to PQ, both being ſines of the ſame arch CP. And becauſe the triangles PpA, $B\delta A$, and alſo $B\delta b$, BCD are ſimilar, the ratio of Pp to Ff, being compounded of Pp to $B\delta$, $B\delta$ to $b\delta$ and $b\delta$ to Ff, is compounded of AP to AB or $2BD$, BD to CD and 3 to 4, which make the ratio of $3AP$ to $8CD$ or $8PQ$. Therefore by the rule the denſity at F is to the denſity at P or C (as $Pp \times PQ$ to $Ff \times FG$) as $3AP \times PQ$ to $8PQ \times FG$, or as AC to $\frac{8}{3}$ FG.

* Art. 461.

465. Hence drawing BH perpendicular to the axis ACI, the denſity at F in the ſuperficial cauſtick, is as its ordinate FG, or as the rectangle under BH and HI. For I find that FG is to BH as HI to $\frac{1}{2}$ IC, which is not worth the trouble of a demonſtration. Hence it appears that the denſity of the rays in the axis at K and A, is infinitely greater than at any finite diſtance from it.

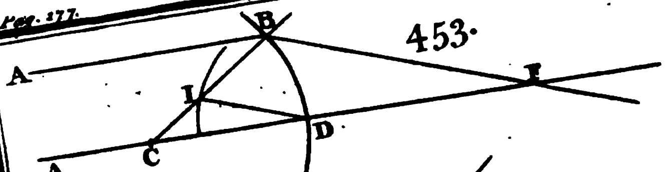
Fig. 443.

466. When the focus A is at an infinite diſtance from the reflecting ſpherical ſurface LBI, the denſity of the rays at any point F of the ſuperficial cauſtick, deſcribed by the revolution of the linear cauſtick LFK about the axis ACI, is to the uniform denſity of the incident rays upon a perpendicular plane CDL, as BD to FG, that is as the coſine of the angle of incidence to the ordinate drawn from the point F. For the portion LEF of the linear cauſtick is equal to $\frac{3}{2}$ BD*; and therefore $Ff = \frac{3}{2} b\delta$. But Pp is to Ff in a ratio compounded of Pp or $B\delta$ to $b\delta$ and of $b\delta$ to Ff, that is of BD to DC and of 2 to 3. Therefore the denſity at F is to the denſity at D as ($Pp \times PQ$ to $Ff \times FG$* that is as) $2BD \times PQ$ to $3CD \times FG$, or as BD to $\frac{3}{2}$ FG.

* Art. 461.

* Art. 463.

467. Hence the denſity at F is as BD directly and FG inverſely; or as BD directly and CD cube inverſely. For I find that FG is to CD as CD^2 to CI^2. Hence it appears that the denſity of the rays at the cuſp K is infinitely greater than when the ordinate FG has a finite magnitude.



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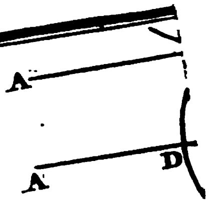

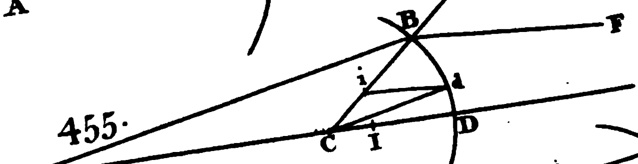

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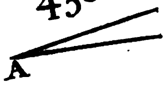

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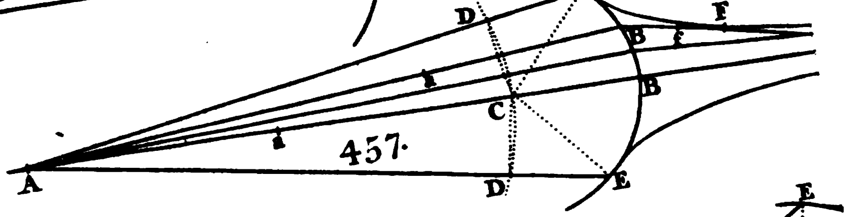

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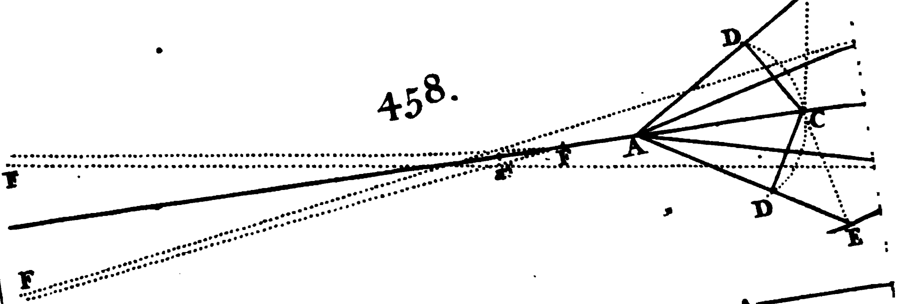

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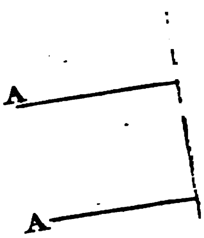

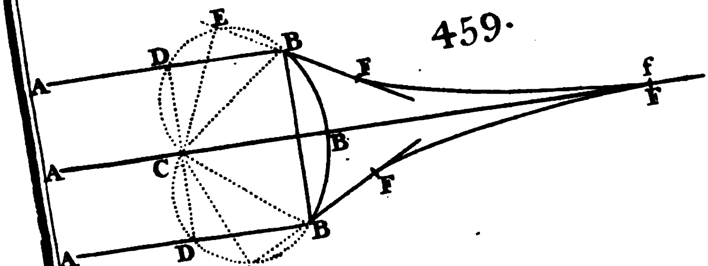

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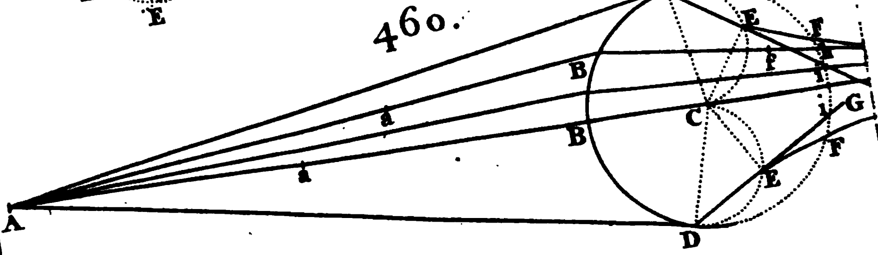

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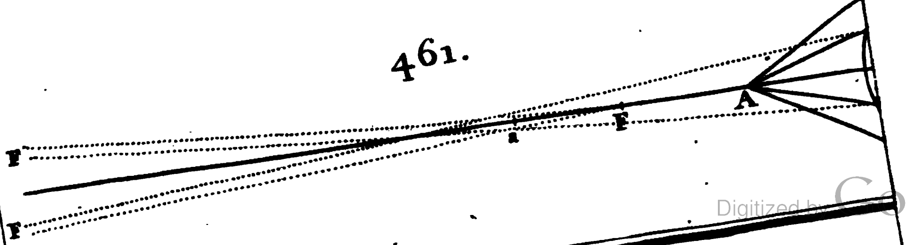

461.

468. By theſe rules we have the proportions of the heat or warmth of the rays in the ſeveral parts of theſe cauſticks, both with reſpect to one another and to that of the incident rays upon a perpendicular ſurface; upon ſuppoſition that the heat of the rays in any ſurface is proportionable to their denſity, whatever be their mutual inclinations to one another.

LEMMA. IV.

469. In the angle of incidence ABC or its complement to two right ones, inſcribe a line AI taken in proportion to AB as the ſine of incidence to the ſine of refraction; and the refracted ray BF will be parallel to AI. Fig. 444, 445.

For in the triangle ABI, the ſine of the angle ABI is to the ſine of the angle AIB as AI to AB*, that is by conſtruction as the ſine of incidence to the ſine of refraction. But ABI is the angle of incidence or its complement to two right ones, and therefore AIB or IBF is the angle of refraction or its complement to two right angles. * Art. 221.

It muſt be obſerved that a circle whoſe center is A and ſemidiameter is AI, will cut BC produced in two points I and i, and conſequently two lines BF, Bf may be drawn from B reſpectively parallel to AI and Ai, making equal angles with CBi on each ſide of B; but it is eaſy to diſtinguiſh which of the lines BF, Bf is deſcribed by the refracted ray, by obſerving whether the refraction be made towards the perpendicular BC or from it. Fig. 446.

470. *Corol.* 1. Hence when the ſurface of the denſer medium is convex in the axis AC take CT to TD as the ſine of incidence to the ſine of refraction; and if CA be greater than CT, all the rays falling on the circle DB will converge towards the diameter CD. For then the ratio of CA to AB will be always nearer to a ratio of equality than the ratio of CT to TD or of IA to AB by conſtruction; and conſequently IA and AB will always be on contrary ſides of the axis AC, and therefore BF will always converge towards it; and will cut it in greater angles while DB grows greater. Fig. 444, 445.

471. *Corol.* 2. But if CA be leſs than CT, let the incident ray AH be to AC as the ſine of refraction to the ſine of incidence, and the refracted ray HF will be parallel to the axis; and all the rays whoſe points of incidence are farther from the axis than H will converge towards the axis and the reſt that are nearer will diverge from it. For in the triangle ACH the ſine of the angle AHC is to the ſine of the angle ACH or of CHF as AC to AH*, or in the ratio of refraction; therefore CHF is the angle of refraction. Now if AB be farther from the axis than AH, then AI and AB muſt be on contrary ſides of the axis, to be in the ſame ratio as AC to AH; and therefore BF being parallel to AI converges towards the axis. Fig. 447, 448. * Art. 221.

But when *AB* comes between *AH* and *AC*, then *AI* muſt do ſo too, and ſo *BF* will diverge from the axis.

Fig. 449. 472. *Corol.* 3. When the incident rays enter a convex ſurface of a denſer medium, erect *CE* perpendicular to the axis *CD*, and let *CE* be to the ſemidiameter *CD* or *CK*, as the ſine of refraction to the ſine of incidence; then draw *EK* parallel to the axis, and let *KL* touch the circle in *K* and cut the axis in *L*; and if *CA* be leſs than *CL*, all the rays which came from *A* will diverge from the axis after refraction. Becauſe the tangent *LK* will be refracted into *KE*. But when the rays go from the denſer medium into the rarer, let *CK*, erected perpendicular to the axis, be Fig. 450. the diameter of a ſemicircle *CEK*, in which inſcribe *CE* in proportion to *CK* as the ſine of incidence to the ſine of refraction; and drawing *KE* cutting the axis in *M*, if *CA* be leſs than *CM*, all the refracted rays will diverge from the axis. For *MK* will be refracted into the tangent *KF*. Fig. 451. The reſt appears from the 2d corol. If in this laſt caſe *A* advances firſt to the center and then ſtill nearer to the ſurface, the rays flowing from the center will emerge unrefracted, and then will diverge from the axis the contrary way.

Fig. 452. 473. *Corol.* 4. It is obſervable, that taking *CA*, *CB*, *CG* continual proportionals in the ratio of the ſine of incidence to the ſine of refraction, and placing *A* and *G* both on the ſame ſide of *C* in the denſer medium, all the refracted rays will diverge accurately from the given point *G*. For the triangles *CAB*, *CBG* are equiangular, having their ſides about the common angle *C* proportionable[a]: and ſo the ſine of the angle of incidence *CBA*, is to the ſine of the angle *CBG* or *CAB*, as the oppoſite ſide *CA* to the oppoſite ſide *CB**, that is by conſtruction in the ratio of the ſines that meaſure the refraction; and conſequently *CBG* is the angle of refraction; and the points *A*, *G* are invariable.

[a] Euc. VI. 8.

* Art. 221.

Fig. 453, 454. 474. *Corol.* 5. If the incident rays as *AB* come parallel to *CD*; in the angle *BCD*, or its complement to two right ones, inſcribe a line *DI* taken in proportion to *DC* as the ſine of incidence to the ſine of refraction; and the refracted ray *BF* will be parallel to *DI*. For in the triangle *DCI* the ſine of the angle *DCI*, is to the ſine of *DIC*, as *DI* to *DC**, that is by conſtruction as the ſine of incidence to the ſine of refraction. But *DCI* is equal to the angle of incidence *ABI* or its complement to two right ones; and therefore *DIC* or *FBC* is the angle of refraction or its complement to two right ones.

* Art. 221.

475. *Corol.* 6. Hence we have a practical method of drawing any number of refracted rays very expeditiouſly, by deſcribing an arch with the center *D* and given ſemidiameter *DI*; by drawing any line *CB* cutting this arch in *I*; by joining *DI* and drawing *BF* parallel to it.

476. *Corol.* 7. Hence while the arch *DB* increaſes, the line *CF* decreaſes. For the triangles *CFB*, *CDI* being equiangular, we have *CF*:*CB*::*CD*:*CI*. Therefore *CF* is reciprocally as *CI*.

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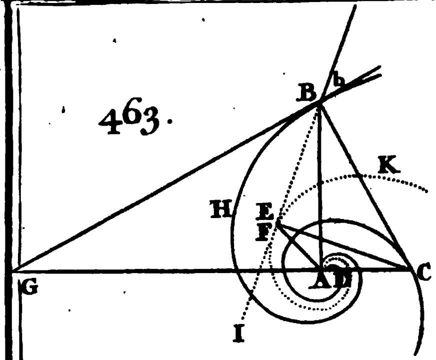

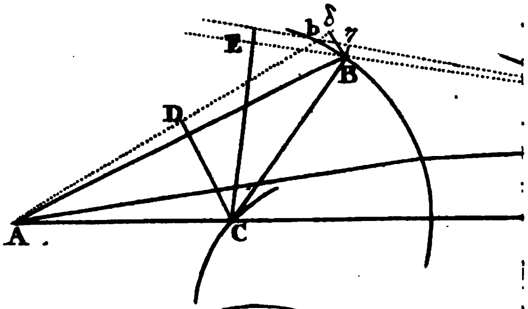

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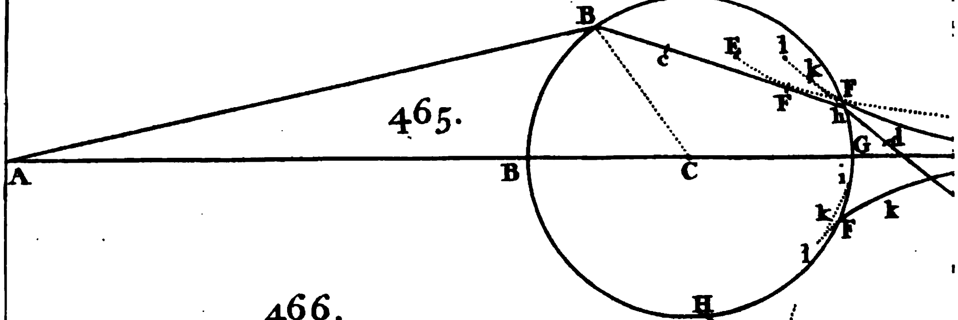

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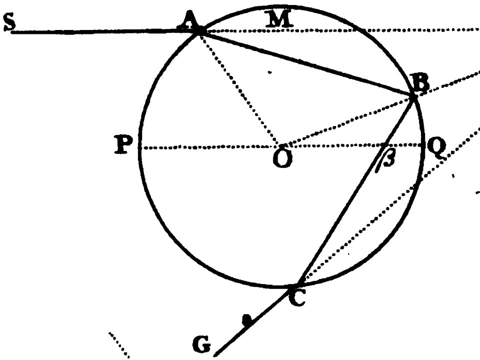

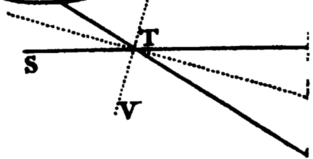

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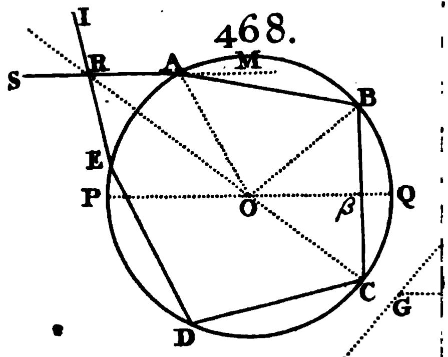

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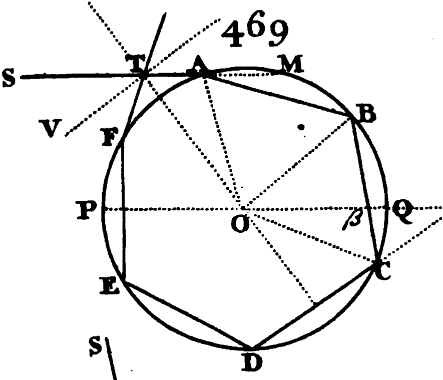

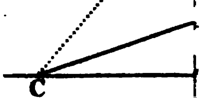

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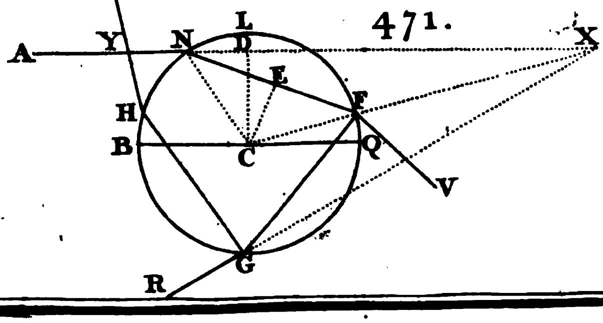

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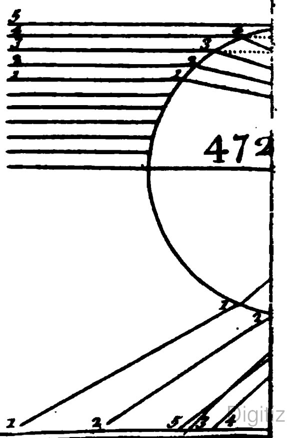

477. *Corol.* 8. When the incident rays diverge from a focus *A*, the refracted rays may be drawn with the like expedition in this manner. Take a line *DI* to *DC* as the ſine of incidence to the ſine of refraction; and draw a ſemidiameter *Cd* parallel to the incident ray *AB*; and in the angle *dCB*, or in its complement to two right ones, inſcribe a line *di* taken equal to the conſtant line *DI*; and draw the refracted ray parallel to *di*. For in the triangle *dCi*, the ſine of the angle *dCi* is to the ſine of *diC*, as *di* to *dC*, as the ſine of incidence to the ſine of refraction, by conſtruction. But the angle *dCi* is equal to the angle of incidence *ABC* or to its complement, and therefore *diC* or *FBC* is the angle of refraction or its complement. Fig. 455, 456.

PROPOSITION XII.

478. *Concerning the ſhapes and properties of cauſticks by refraction.*

Having determined the poſition of any refracted ray, by the foregoing lemma and its corollaries, and alſo the point in that ray where the neareſt ray will interſect it, by art. 423; the points of a cauſtick are thereby determined[a]. But to form an idea of their ſhapes it is neceſſary to conſider ſeveral caſes.

479. *Caſe* 1. Let the focus *A* and the center *C* of a refracting circle *EBE* be both in the denſer medium. Upon the diameter *AC* having drawn a circle *ADCD*, inſcribe in it the equal chords *CD*, *CD*, taken ſeverally in proportion to the whole ſine *CB* or *CE*, as the ſine of incidence to the ſine of refraction; and drawing the incident rays *ADE*, *ADE*, the legs of the cauſtick will begin from *E* and *E*[*], where they touch the refracting circle, and will approach towards each ſide of the axis *ACF* till they meet it in the principal focus *F*, and there form a cuſp or ſharp point; provided *A* be farther from the center than *a* the focus of rays that come the contrary way parallel to *CA*. But if *A* advances to *a*, the diſtance *fF* will be infinite and ſo the axis *ACF* will become an aſymptote to the legs of the cauſtick. And if *A* advances nearer than *a*, the legs will open and have two aſymptotes *BF*, *BF* to which the rays emerging from *B* will run parallel. This muſt neceſſarily happen in a certain poſition of *BA*, that is when rays coming the contrary way parallel to *FB*, are collected to *A*. For the focal diſtance of parallel rays decreaſes as the point of incidence recedes from the axis till it equals *ED*[*]. There are alſo two other imaginary legs belonging to the ſame aſymptotes, that extend themſelves from the focus *F*, now ſituated on the other ſide of the center. Fig. 457, 458. Fig. 458.

480. *Caſe* 2. When *CA* is to *CB* as the ſine of incidence to the ſine of refraction the cauſtick will be contracted into a point *F* from which all the rays will diverge by art. 473.

481. *Caſe* 3. The 459th figure repreſents a piece of a cauſtick made by a thick plano-convex lens *BBB* when a pencil of parallel rays falls per-

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[a] Art. 446.

[*] Art. 420. 423.

[*] Art. 420.

pendicularly upon its plane ſurface; and conſequently are only refracted by the ſpherical ſurface; the poſition of the refracted rays that fell upon the circumference of the lens is determined as in the next caſe.

Fig. 460, 461, 462.

482. *Caſe* 4. While C remains in the denſer medium, tranſpoſe the focus A into the rarer, and draw the tangents AD, AD to the refracting circle DBD and joyn CD, CD upon which as diameters deſcribe the ſemicircles CED, CED towards the denſer medium, and inſcribe in them the lines CE, CE, taken in proportion to the whole ſine CD, as the ſine of refraction to the ſine of incidence. And the legs of the cauſtick beginning from E* in the direction DE, DE will approach towards the axis AC, till they meet in the principal focus F; provided CA be greater than Ca; or will have the ſame poſitions as in the firſt caſe.

* Art. 420. 423.

Fig. 460.

483. The focus A being in the rarer medium, let the circle DBD be continued quite round, and let it cut the cauſtick in F and F, the axis AC in G and any other ray ABF in b; and while the ray AB is carried with an angular motion round about A from the axis AC towards the tangent AD, the arch Gb will firſt increaſe till it equals the arch GF, and then will decreaſe again till it equals the arch Gi, cut off by the laſt refracted ray DEi. This is manifeſt from the motion of the refracted ray BbF while it touches the convexity of the cauſtick in F, provided the focus A be ſo remote from the ſurface that the laſt refracted ray DEi may converge towards the axis AC.

Fig. 463.

484. Cauſticks made by refractions at other curves are alſo determinable by the 423d article. For inſtance, the imaginary cauſtick AFK made by refractions at an equiangular ſpiral AHB is alſo an equiangular ſpiral, their common pole A being the focus of incident rays. For ſuppoſing what has already been ſaid of this curve[a]; from the center C of a circle of equal curvity to it at B, let down the ſines of incidence and refraction CD, CE upon the incident and refracted rays AB and BFI produced backwards. And becauſe D coincides with the focus A of incident rays, E does alſo coincide with F the focus of refracted ones[b]. Joyn AE or AF, and becauſe the angular points A and E of the right angles CAB, CEB are in a ſemicircumference whoſe diameter is CB*, the angles CBA, CEA ſtanding upon the ſame chord CA are equal[c] and being ſubſtracted from the right angles CBG, CEI, the remaining angles ABG, AEI, which the lines AB, AE or AF do make with the curves, are every where equal; which is the property of the ſpiral. So that this cauſtick-ſpiral differs from the other in poſition only.

[a] Art. 459.

[b] Art. 418, 423.

* Euc. III. 35.

[c] Euc. VI. 23.

Fig. 464.

485. To find the length of any cauſtick by refractions, imagine the reflected ray BF produced, to be unlaped like a ſtring from the convexity of the cauſtick $F\varphi$; then becauſe the figures $b\delta\eta B$, $CDEB$ are ſimilar[d], the increment $b\delta$ of the incident ray AB, is every where to the decrement $b\eta$ of the ſtring $BF\varphi$, in the given ratio of the ſine CD to the ſine CE;

[d] Art. 421 demonſtr.

(for which put n to m;) and therefore when AB, BF come into a new poſition $A\beta$, $\beta\varphi$, the ſum of the increments of AB, that is $A\beta-AB$, is to the ſum of the decrements of the ſtring, that is $BF+F\varphi-\beta\varphi$, as n to m. Whence $\frac{m}{n}\times\overline{A\beta-AB}=BF+F\varphi-\beta\varphi$, or $\frac{m}{n}\times\overline{A\beta-AB}+\beta\varphi-BF=F\varphi$.

Fig. 465.

486. To find the points of any cauſtick made by two ſucceſſive refractions, let the ray BFb which touches the cauſtick EFF in F, (made as before by the firſt refraction of the rays), meet with another refracting curve GbF, or with the ſame curve continued; and let it be refracted at b into the line bd; in which let bd be the focal diſtance of other rays coming parallel to Bb, and in bB let bc be the focal diſtance of other rays coming parallel to db; then ſince F is the focus of incident rays upon the curve bG, ſay as $Fc:cb::bd:dk$, and placing dk as uſual[a], the point k will be the focus of a ſlender pencil after both refractions, or a point of the ſecond cauſtick KFk; whoſe points may alſo be found by art. 434 and 436, without finding the points F of the firſt cauſtick.

[a] Art. 423.

Fig. 465.

487. Hence it will appear that a cauſtick made by refractions through a circular ſection of a cylinder or a great circle of a ſphere, will have ſuch a ſhape as is here repreſented. Each half of this cauſtick on each ſide of the axis ACK conſiſts of two arches $KkFl$ and lki, that are convex towards one another and form a cuſp at l within the circle. The arch $KkFl$ of the ſecond cauſtick cuts the circle in the ſame point F as the firſt cauſtick does. For by the proportion above, when the points F, b coincide, or when Fc equals cb, then bd and dk are alſo equal. The reaſon of the cuſp at l is this; that while bk is increaſing, and then decreaſing again, the point b is continually approaching towards G *. The arches KFl and lki are convex towards one another, becauſe the emergent ray, while its point of contact k is moving from K to F, to l and to i, cuts the axis CK in greater and greater angles, till at laſt it emerges at i in a tangent to the circle and to the cauſtick too. When the focus of incident rays is nearer to the ſphere than its focal diſtance, the ſecondary cauſtick FKk will have two aſymptotes like as the primary one; and their ſhapes will be much alike.

* Art. 483.

CHAPTER X.

Concerning the Rain-bow.

PROPOSITION I.

488. WHEN *a ray of light is refracted into a circle, and ſucceſſively reflected within it any given number of times before it emerges out of the circle by a ſecond refraction; let the angle of refraction be multiplyed by the number of ſucceſſive reflections increaſed by an unite; and the exceſs of the reſulting angle above the angle of incidence will be equal to half the angle contained under the incident and the emergent ray produced till they meet: that is, the exceſs abovementioned is equal to half that angle, under the incident and the emergent ray, in which the refracting circle lyes, when the number of reflections is odd; and is equal to half the other angle, under the ſame rays, which is the complement of the former to two right angles, when the number of reflections is even.*

Fig. 466 to 469. For let $ABCDE$ be a great circle of a ſphere whoſe center is O, and let an incident ray SA be refracted at A to B, and be reflected from B to *Art. 183 &c. C; and at C let it either go out by refraction to G, or be reflected to D*; where let it either go out by refraction to H or be reflected to E; and ſo on. And when the number of reflections is odd, a line OR drawn through the center O and the middlemoſt point of reflection, will biſect the angle at R under the incident and the emergent ray produced: becauſe the reflections and refractions on each ſide of the line OR are equal in number and magnitude; the chords AB, BC, CD, DE deſcribed by the reflected ray being equal to one another. And for the ſame reaſon when the number of reflections is even, a line OT, drawn through the center O perpendicular to the chord that joins the two middlemoſt points of reflection, will biſect one of the angles at T under the incident and the emergent ray produced; and a line TV, perpendicular to TO, will biſect the other angle under them, which is the complement of the former to two right ones. Hence the line TV is parallel to the middlemoſt chord, becauſe TO is perpendicular to them both. Draw a diameter POQ parallel to the incident ray SAM, and let it cut the reflected rays BC, CD, DE produced, in β, γ, δ, reſpectively. Join OA, OB and in fig. 466 the ſums of the three angles in each of the triangles OAB, OAR, are equal to one another; take away the common angle AOB, and the ſum of the equal angles OAB, OBA in the firſt triangle, will be equal to the ſum of the angles OAR, ORA in the ſecond triangle. And by ſubſtracting the angle of incidence OAR or OAM from both ſums, we have $2OAB-OAM=ORA=BOQ$. Hence in fig. 467. the angle STV or $P\beta C$, being an external angle of the triangle $OB\beta$, equals $OBC + BOQ = OAB + 2OAB - OAM = 3OAB - OAM$. Hence again in fig. 468 the angle SRO or POC, being an external angle of the triangle $OC\beta$, equals $OCB + P\beta C = OAB + 3OAB - OAM = 4OAB - OAM$. Hence again in fig. 469 the angle STV or $P\gamma D$, being an internal angle of the triangle $CO\gamma$, equals $OCD - CO\gamma = 5OAB - OAM$, throwing away two right angles. For $CO\gamma = 2$ right angles $- POC = 2$ right angles $- 4OAB + OAM$. And ſo forward continually. Therefore if the number of ſucceſſive reflections increaſed by an unite be called m, it appears that $mOAB - OAM$ equals half the angle under the incident and emergent rays, *Q. E. D.*

PROPOSITION II.

489. *Things remaining as they were, let the angle of incidence increaſe from nothing till it becomes a right angle; and the angle under the incident and the emergent ray, after any given number of reflections called* n, *will firſt increaſe and then decreaſe again; and will be the greateſt of all when the tangent of the angle of incidence, is to the tangent of the angle of refraction, as* $n + 1$ *to* 1.

For putting $m = n + 1$, we had half the angle under an incident and the emergent ray equal to the exceſs of $mOAB$ above OAM*; which exceſs, when the angles OAB, OAM are very ſmall, will alſo be but ſmall; and will increaſe ſo long as the ſucceſſive increments of $mOAB$ ſhall exceed the contemporary increments of OAM; and will decreaſe again when the ſucceſſive increments of $mOAB$ are exceeded by the increments of OAM; and conſequently will be the greateſt of all when m times the leaſt increment of OAB is equal to once the contemporary increment of OAM; that is when the leaſt increment of the angle of incidence OAM is to the contemporary increment of the angle of refraction OAB, and conſequently the tangent of incidence is to the tangent of refraction[a], as m to 1. *Q. E. D.*

Fig. 466 to 469.

* Art. 448.

[a] Art. 421.

PROPOSITION III.

490. *It is propoſed to find two angles, whoſe ſines ſhall be in a given ratio of* I *to* R, *and whoſe tangents ſhall be in another given ratio of* m *to* 1.

In any given line $CEDA$, let CA be to CD as I to R, and CA to CE as m to 1; with the center C and ſemidiameter CD deſcribe an arch DB, cutting a circle ABE whoſe diameter is AE, in B; draw ABF, and joining BC, the ſine of the angle CBF will be to the ſine of CAF as I to R; and the tangent of CBF to the tangent of CAF as m to 1; and conſequently CBF, CAF are the angles required. For in the triangle CAB the ſine of the angle CBA or CBF, is to the ſine of CAF, as CA to CB* Fig. 470.

* Art. 221.

or CD, as I to R by conſtruction. Join BE and compleat the parallelogram $CEBG$; and CG produced will cut ABF at right angles in F, becauſe ABE is a right angle in the ſemicircle ABE. Therefore the lines FC, FG are tangents of the angles CBF, GBF or CAF to the radius BF; * Euc. VI. 2. and the tangent FC is to the tangent FG as FA to FB* or as CA to CE* or as m to 1 by conſtruction. *Q. E. D.*

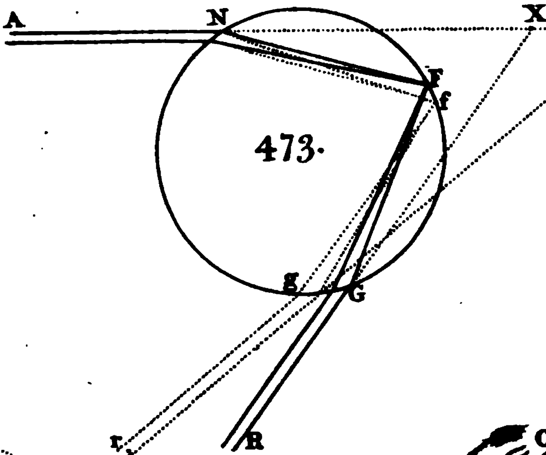
491. *Corol.* 1. When parallel rays of the ſun fall upon a ſpherical drop of rain, let the given ratio of I to R ſtand for the ratio of the ſine of incidence to the ſine of refraction; and let n be any given number of ſucceſſive reflections made by every ray before it emerges out of the drop, and let $m=n+1$; and by theſe propoſitions it appears, that half the greateſt angle which any of the emergent rays can make with the incident rays, is equal to $m\times\text{ang.}\,CBF-CAF$. For CBF and CAF or GBF are angles whoſe ſines are as I to R, and whoſe tangents are as m to 1; and conſequently are the angles of incidence and refraction of that ray, whoſe incident and emergent parts produced contain the greateſt angle.

492. *Corol.* 2. The foregoing conſtruction for determining the angle a Phil. Tranſ. N°. 297. CBF is Dr. *Halley*'s[a], and Sir *Iſaac Newton*'s rule for calculating it, is this that follows. As $\sqrt{\overline{mm-1}\times RR}$ is to $\sqrt{\overline{II-RR}}$, ſo is the tabular radius to the coſine of the angle of incidence CBF. Whence this angle and its ſine are given by the tables, and from thence by the ratio of I to R the tabular ſine of the angle of refraction and the angle it ſelf are alſo given. The rule may thus be demonſtrated. We had $CA:CB::I:R$ and $FA:FB::m:1$. Hence $CA^2=\frac{II}{RR}CB^2$, and $AF^2=mmBF^2$; and ſo * Euc. I. 47. $\frac{II}{RR}CB^2-mmBF^2=(CA^2-AF^2=FC^2\,{*}=)\,CB^2-BF^2$. Hence $\frac{II}{RR}CB^2-CB^2=mmBF^2-BF^2$, and $\overline{II-RR}\times CB^2=\overline{mm-1}\times RR\times BF^2$; and by reſolving this equality into a proportion, and by extracting the roots, we have $\sqrt{\overline{mm-1}\times RR}:\sqrt{\overline{II-RR}}::CB:BF::$ radius : coſine ang. CBF.

PROPOSITION IV.

To explain the Phænomena of the Rain-bow.

Deſign. 493. Having premiſed ſuch mathematical principles as are neceſſary for an exact computation of the apparent diameters and breadths of the Rain-bows, I will here ſubjoin Sir *Iſaac Newton*'s entire explication of the colours of the bows and of the manner in which they are formed; taking the liberty here and there of making a few additions to it; for the ſake of ſuch readers as may not be ſo ſkilful as thoſe that he generally writes to.



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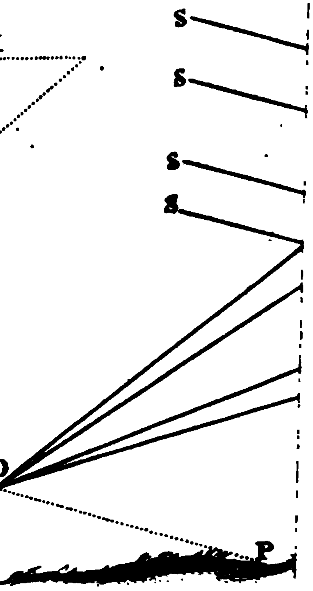

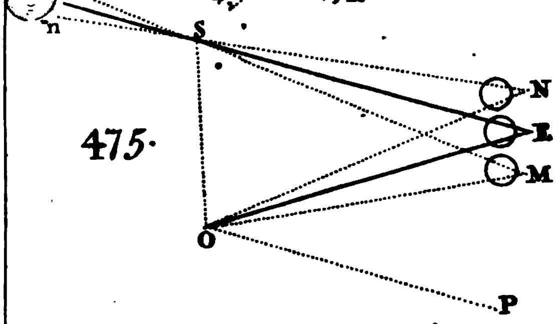

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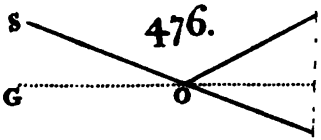

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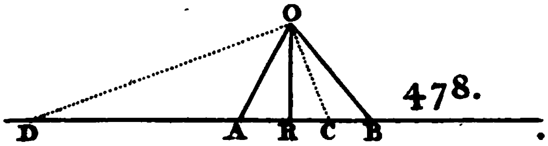

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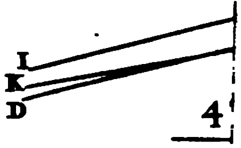

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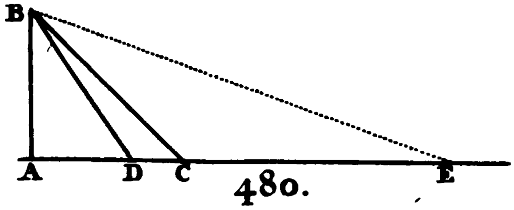

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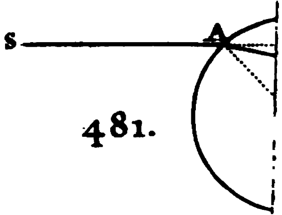

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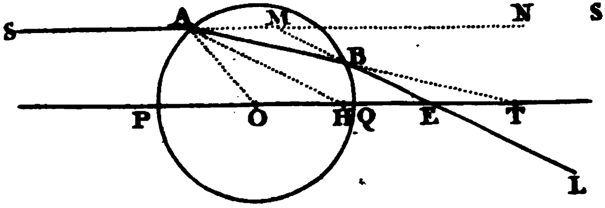

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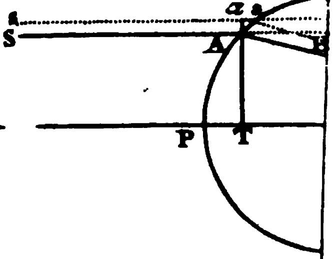

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Newt. Opt. p. 147.

494. This bow never appears but where it rains in the ſun-ſhine, and may be made artificially by ſpouting up water which may break aloft, and ſcatter into drops, and fall down like rain. For the ſun ſhining upon theſe drops, certainly cauſes the bow to appear to a ſpectator ſtanding in a due poſition to the rain and ſun. And hence it is now agreed upon that this bow is made by refraction of the ſun's light in drops of falling rain. This was underſtood by ſome of the ancients, and of late more fully diſcovered and explained by the famous *Antonius de Dominis* Archbiſhop of *Spalato* in his book *de Radiis Viſus & Lucis* publiſhed by his friend *Bartolus* at *Venice* in the year 1611, and written above 20 years before. For he teaches there how the interior bow is made in round drops of rain by two refractions of the ſun's light, and one reflection between them; and the exterior bow by two refractions and two ſorts of reflections between them in each drop of water; and proves his explications by experiments made with a phial full of water, and with globes of glaſs filled with water, and placed in the ſun to make the colours of the two bows appear in them. The ſame explication *Des-Cartes* has purſued in his *Meteors* and mended that of the exterior bow. But while they underſtood not the true origin of colours, it is neceſſary to purſue it a little farther.

Fig. 471.

495. For underſtanding therefore how the bow is made, let a drop of rain or any other ſpherical tranſparent body be repreſented by the ſphere *BNFG* deſcribed with the center *C* and ſemidiameter *CN*. And let *AN* be one of the ſun's rays incident upon it at *N*, and be thence refracted to *F*, where let it either go out of the ſphere by refraction toward *V*, or be reflected to *G*; and at *G* let it either go out by refraction to *R*, or be reflected to *H*; and at *H* let it go out by refraction towards *S*, cutting the incident ray in *Y*. Produce *AN* and *RG* till they meet in *X*, and upon *AX* and *NF* let fall the perpendiculars *CD* and *CE*, and produce *CD* till it falls upon the circumference at *L*. Parallel to the incident ray *AN* draw the diameter *BQ*; and let the ſine of incidence out of air into water, be to the ſine of refraction as *I* to *R*. Now if you ſuppoſe the point of incidence *N* to move from the point *B* continually till it comes to *L* the arch *QF* will firſt increaſe and then decreaſe, and ſo will the angle *AXR* which the rays *AN* and *GR* contain; and the arch *QF* and angle *AXR* will be biggeſt when *ND* is to *NC* as $\sqrt{II-RR}$ to $\sqrt{3RR}$, in which caſe *NE* will be to *ND* as 2*R* to *I*. Alſo the angle *AYS*, which the rays *AN* and *HS* contain will firſt decreaſe and then increaſe; and grow leaſt when *ND* is to *NC* as $\sqrt{II-RR}$ to $\sqrt{8RR}$, in which caſe *NE* will be to *ND* as 3*R* to *I*; and ſo the angle which the next emergent ray (that is the emergent ray after three reflections) contains with the incident ray *AN* will come to its limit, when *ND* is to *NC* as $\sqrt{II-RR}$ to $\sqrt{15RR}$; in which caſe *NE* will be to *ND* as 4*R* to *I*. And the angle

which the ray next after that emergent, (that is the ray emergent after four reflections) contains with the incident, will come to its limit, when ND is to NC as $\sqrt{II-RR}$ to $\sqrt{24RR}$; in which caſe NE will be to ND as $5R$ to I; and ſo on infinitely, the numbers 3, 8, 15, 24, &c. being gathered by continual addition of the terms of the arithmetical progreſſion 3, 5, 7, 9, &c. The truth of all which is evident either by art. 492. or by art. 429, 430.

496. Now it is to be obſerved, that as when the ſun comes to his tropicks, days increaſe and decreaſe but a very little for a great while together; ſo when by increaſing the diſtance CD, theſe angles come to their limits, they vary their quantity but very little for ſome time together; and therefore a far greater number of rays which fall upon all the points N in the quadrant BL ſhall emerge in the limits of theſe angles than in any other inclinations. Fig. 472. Add to this, that of all the rays which fall upon the quadrant BL, thoſe contiguous ones can only emerge parallel to one another, which emerge in the limits of theſe angles; and that all other contiguous rays emerge diverging from points either behind or before the drop; and conſequently will fall much thinner upon the eye, at a great diſtance from the drop, than the parallel rays. For thoſe rays which converge to points behind the eye, placed at a great diſtance from a ſmall drop, are not ſenſibly different from parallel rays. This will appear by obſerving that while the arch BN is continually increaſing from nothing, and the angle AXR, for example, is alſo increaſing; the ſucceſſively emergent rays, being continually leſs and leſs inclined to the incident ones or to the fixt line BQ, are alſo ſucceſſively inclined in ſmall angles to one another; and the ſame thing is manifeſt while the angle AXR is decreaſing; the ſucceſſive rays being more and more inclined to PQ; conſequently in the limit between the increaſe and decreaſe of this angle the contiguous incident rays muſt emerge parallel to one another.

497. And farther it is to be obſerved, that the rays which differ in refrangibility will have different limits of their angles of emergence; and by conſequence according to their different degrees of refrangibility, emerge moſt copiouſly in different angles; and being ſeparated from one another appear each in their proper colours. Add to this that although the heterogeneal rays of any ſlender pencil whatever, as AN, will be ſeparated by refractions at the drop into rays $NFGR$ of one colour, and Fig. 473. $Nfgr$ of another, as by refractions through a priſm; yet theſe emergent rays GR, gr will not affect the eye with their diſtinct colours, unleſs they be in the limits of the angles AXR, Axr; becauſe every where within theſe greateſt angles, an infinite number of ſuch coloured pencils being variouſly inclined to one another are mixt together, and conſequently appear white or without diſtinct colours. And the ſame may be ſaid of the rays emerging any where within the greateſt angle NYS. Fig. 471.

498. Now what theſe angles are may be gathered firſt by computing the angles of incidence and refraction by art. 492 or 429 and 430, and then the angles *AXG*, *AYS* themſelves by the 488th article. For in the leaſt refrangible rays the ſines *I* and *R* are 108 and 81 *, and thence by computation the greateſt angle *AXR* will be found 42 degrees and 2 minutes; and the leaſt angle *AYS*, 50 degrees and 57 minutes. And in the moſt refrangible rays the ſines *I* and *R* are 109 and 81, and thence by computation the greateſt angle *AXR* will be found 40 degrees and 17 minutes, and the leaſt angle *AYS*, 54 degrees and 7 minutes.

* Art. 179.

Fig. 474.

499. Suppoſe now that *O* is the ſpectator's eye, and *OP* a line drawn parallel to the ſun's rays. Let *POE*, *POF*, *POG*, *POH* be angles of 40 degr. 17 min.; 42 deg. 2 min.; 50 deg. 57 min.; 54 deg. 7 min. reſpectively; and theſe angles turned about their common ſide *OP*, ſhall with their other ſides *OE*, *OF* and *OG*, *OH* deſcribe the verges of two rain-bows *AFBE* and *CHDG*. For if *E*, *F*, *G*, *H* be drops placed any where in the conical ſuperficies deſcribed by *OE*, *OF*, *OG*, *OH* and be illuminated by the ſun's rays *SE*, *SF*, *SG*, *SH*; the angle *SEO* being equal to *PEO* or 47 deg. 17. min. ſhall be the greateſt angle in which the moſt refrangible rays can after one reflection be refracted to the eye; and therefore all the drops in the line *OE* ſhall ſend the moſt refrangible rays moſt copiouſly to the eye; and thereby ſtrike the ſenſes with the deepeſt violet colour in that region. In like manner the angle *SFO* being equal to the angle *POF* or 42 deg. 2 min. ſhall be the greateſt in which the leaſt refrangible rays after one reflection can emerge out of the drops; and therefore theſe rays ſhall come moſt copiouſly to the eye from the drops in the line *OF*, and ſtrike the ſenſes with the deepeſt red colour in that region. And by the ſame argument the rays which have intermediate degrees of refrangibility ſhall come moſt copiouſly from drops between *E* and *F* and ſtrike the ſenſes with the intermediate colours in the order which their degrees of refrangibility require; that is in the progreſs from *E* to *F* or from the inſide of the bow to the outſide in this order, violet, indigo, blue, green, yellow, orange, red. But the violet by the mixture of the white light of the clouds will appear faint and incline to purple. It may be farther obſerved, that all the rays but the violet in the line *SE* will emerge from *E* in a greater angle than *SEO* made by the violet, and conſequently will paſs below the eye; and that all the rays but the red in the line *SF* will emerge from *F* in a leſſer angle than *SFO* made by the red, and conſequently will paſs above the eye; by which means only red will appear in the line *SF* and only violet in the line *SE*.

500. Again the angle *SGO* being equal to the angle *POG* or 50 deg. 51 min. ſhall be the leaſt angle in which the leaſt refrangible rays can after two reflections emerge out of the drops; and therefore the leaſt refrangible rays ſhall come moſt copiouſly to the eye from the drops in the

line *OG*, and ſtrike the ſenſe with the deepeſt red in that region. And the angle *SHO* being equal to the angle *POH* or 54 deg. 7 min. ſhall be the leaſt angle in which the moſt refrangible rays after two reflections can emerge out of the drops; and therefore theſe rays ſhall come moſt copiouſly to the eye from the drops in the line *OH*, and ſtrike the ſenſe with the deepeſt violet in that region. And by the ſame argument the drops in the regions between *G* and *H* ſhall ſtrike the ſenſe with intermediate colours in the order which their degrees of refrangibility require; that is in the progreſs from *G* to *H*, or from the inſide of the bow to the outſide in this order; red, orange, yellow, green, blue, indigo, violet. And ſince theſe four lines *OE*, *OF*, *OG*, *OH* may be ſituated any where in the abovementioned conical ſurfaces, what is ſaid of the drops and colours in theſe lines is to be underſtood of the drops and colours every where in thoſe ſurfaces.

501. Thus ſhall there be made two bows of colours, an interior and ſtronger by one reflection in the drops, and an exterior and fainter by two; for the light becomes fainter by every reflection. And their colours ſhall lye in a contrary order to one another; the red of both bows bordering upon the ſpace *GF*, which is between the bows. The apparent breadth of the interior bow *EOF*, meaſured croſs the colours, ſhall be 1 deg. 45 min. and the breadth of the exterior, *GOH*, ſhall be 3 deg. 10 min. and the apparent diſtance between them, *GOF*, ſhall be 8 deg. 55 min. the greateſt ſemidiameter of the innermoſt, that is, the angle *POF* being 42 deg. 2 min. and the leaſt ſemidiameter of the outermoſt, *POG*, being 50 deg. 57 min.

502. Theſe are the meaſures of the bows as they would be were the ſun but a point, for by the breadth of his body the breadths of the bows will be increaſed and their diſtance decreaſed by half a degree. And ſo the breadth of the interior Iris will be 2 deg. 15 min., that of the exterior 3 deg. 40 min., their diſtance 8 deg. 25. min., the greateſt ſemidiameter of the interior bow 42 deg. 17 min., and the leaſt of the exterior 50 deg. 42 min. For let *SEO* be the limit of all the angles under the rays of any one colour, which coming from the center of the ſun are reflected from the drop at *E* to the eye at *O*. In the ray *SE* take any point *S* at pleaſure and let the angles *ESM*, *ESN* and alſo *EOM*, *EON* be ſeverally equal to a quarter of a degree, that is to half the apparent breadth of the ſun. And joining *OS*, ſince the ſums of the angles at the baſe *OS*, of the ſeveral triangles *OSM*, *OSE*, *OSN*, are equal among themſelves, their vertical angles at *M*, *E*, *N* are alſo equal among themſelves. Conſequently the angle *SMO* will be the limit of all the angles contained under the incident and emergent rays of the ſame colour as before, which came from *m* the higheſt point of the ſun; and *SNO* the limit of all the angles contained under the incident and emergent rays of the ſame colour as before, which came from *n* the loweſt point of the ſun. Therefore if all the rays of the ſun were of the ſame colour, or alike

Fig. 475.

alike refrangible, the apparent breadth of the bow, meaſured by the angle *MON*, would be but half a degree or equal to the apparent breadth of the ſun meaſured by the angle *MSN* or *mSn*. But ſince his rays are differently refrangible, conceive the drop *E* to be placed any where in the inward or outward verges of the bows, above deſcribed upon ſuppoſition that the ſun was but a point; and then it is manifeſt that the angle *EOM* muſt be added to the inſide, and *EON* to the outſide of the angles which the breadths of thoſe bows ſubtend at *O*, to obtain their apparent breadths. A rain-bow is therefore a circular image of the ſun reflected to the eye from the farther ſurfaces of innumerable drops of falling rain, and dilated in breadth by the unequal refrangibility of rays of different colours.

503. And ſuch are the dimenſions of the bows in the heavens found to be very nearly, when their colours appear ſtrong and perfect. For once by ſuch means as I then had I meaſured the greateſt ſemidiameter of the interior iris about 42 degrees, the breadth of the red, yellow, green in that iris 63 or 64 minutes, beſides the outmoſt faint red obſcured by the brightneſs of the clouds, for which we may allow 3 or 4 minutes more. The breadth of the blue was about 40 minutes more beſides the violet, which was ſo much obſcured by the brightneſs of the clouds that I could not meaſure its breadth. But ſuppoſing the breadth of the blue and violet together to equal that of the red, yellow and green together; the whole breadth of this iris will be about $2\frac{1}{4}$ degrees, as above. The leaſt diſtance between this iris and the exterior iris was about 8 degrees and 30 minutes. The exterior iris was broader than the interior, but ſo faint, eſpecially on the blue ſide that I could not meaſure its breadth diſtinctly. At another time when both bows appeared more diſtinct I meaſured the breadth of the interior iris 2 deg. 10 min. and the breadth of the yellow and green in the exterior iris was to the breadth of the ſame colours in the interior as 3 to 2.

504. Whoever has a mind to repeat theſe obſervations after Sir *Iſaac Newton* may obſerve, that the apparent ſemidiameter of the bow, (or of any ring of colours in either of the bows) is equal to the apparent altitude of its higheſt point added to the ſun's altitude, and conſequently may be meaſured by a common quadrant. For let *SOP* be the axis of the bows Fig. 476. paſſing through the ſun at *S* and the eye at *O*, *GOH* an horizontal line, *E* the higheſt point of any ring of either of the bows, whoſe apparent ſemidiameter *EOP* is required. It is manifeſt that the angle $EOP = EOH + HOP = EOH + SOG$.

505. This explication of the rain-bow is yet farther confirmed by the known experiment (made by *Antonius de Dominis* and *Des-Cartes*) of hanging up any where in the ſun-ſhine a glaſs-globe filled with water, and viewing it in ſuch a poſture that the rays which come from the globe to the eye may contain with the ſun's rays an angle of either 42 or 50 degrees.

grees. For if the angle be about 42 or 43 degrees the ſpectator ſuppoſed at *O*, ſhall ſee a full red colour in that ſide of the globe which is oppoſed to the ſun; as it is repreſented at *F*: and if that angle be made leſs, ſuppoſe by depreſſing the globe to *E*, there will appear other colours yellow, green and blue ſucceſſively in the ſame ſide of the globe. But if the angle be made about 50 degrees, ſuppoſe by lifting up the globe to *G*, there will appear a red colour in that ſide of the globe which lyes toward the ſun: and if the angle be made greater, ſuppoſe by lifting up the globe to *H*, the red will turn ſucceſſively to other colours, yellow, green and blue. The ſame thing I have tryed by letting a globe reſt, and by raiſing or depreſſing the eye, or otherwiſe moving it to make the angle of a juſt magnitude. So far Sir *Iſaac Newton*.

506. It remains now to take notice of ſeveral ſlender rings of colours contiguous to the inſide of the firſt rain-bow These have been obſerved and particularly deſcribed by the Reverend Dr. *Langwith* in the Philoſophical Tranſactions N°. 375. His beſt obſervation was this. The colours of the primary rain-bow were as uſual, only the purple very much inclining to red and well defined. Under thiswas an arch of green, the upper part of which inclined to a bright yellow, the lower to a more duſky green: under this were alternately two arches of rediſh purple and two of green: under all a faint appearance of another arch of purple which vaniſhed and returned ſeveral times ſo quick that we could not fix our eyes upon it. Thus the orders of the colours were.

I. Red, orange, yellow, green, light blue, deep blue, purple.
II. Light green, dark green, purple.
III. Green, purple.
IV. Green, faint vaniſhing purple.

Thus we had four orders of colours and perhaps the beginning of a fifth; for I make no queſtion but that which I call purple (when very red) is a mixture of purple of the upper ſeries with the red of the next below it: and the green a mixture of the intermediate colours. There are two things, which will deſerve to be taken notice of, as they may perhaps direct us to the ſolution of this curious phænomenon. Firſt that the breadth of the firſt ſeries ſo far exceeded that of all the reſt, that as near as I could judge, it was equal to them all taken together. Secondly that I have never obſerved theſe inner orders of colours in the lower parts of the rain-bow, though they have been incomparably more vivid than the upper parts, under which theſe colours have appeared. I have taken notice of this ſo very often that I can hardly look upon it to be accidental; and if it ſhould prove true in general, it will bring the diſquiſition into a narrow compaſs. For it will ſhew that this effect depends upon ſome property, which the drops retain while they are in the upper part of the air, but loſe as they come lower, and are more mixed with one another. I am of opinion that the rain-bow ſeldom appears very lively, without ſome of thoſe orders of colours; and that the ſuppoſed exact agreement between the colours of the rain-bow and thoſe of the priſm, is the reaſon that it has been ſo little obſerved. So far Dr. *Langwith*. To which Dr. *Pemberton* ſubjoins the following theory, printed in the Tranſactions N°. 375. and in his view of Sir *Iſaac Newton*'s Philoſophy p. 401.

507. Sir *Iſaac Newton* has obſerved that in glaſs which is poliſhed and quick-ſilvered, there is an irregular reflection made, whereby ſome ſmall quantity of light is ſcattered from the principal reflected beam. If we allow the ſame thing to happen in the reflection whereby the rain-bow is cauſed, it ſeems ſufficient to produce the appearance abovementioned. Let *AB* repreſent a globule of water, *B* the point from which the rays of any determinate ſpecies, being reflected to *C* and afterwards emerging in the line *CD*, would proceed to the eye and cauſe the appearance of the colour in the rain-bow, which appertains to this ſpecies. Here ſuppoſe that beſides what is reflected regularly, ſome ſmall part of the light is irregularly ſcattered every way, ſo that from the point *B* beſides the rays that are regularly reflected from *B* to *C*, ſome ſcattered rays will return in other lines, as in *BE*, *BF*, BG, *BH*, on each ſide the line *BC*. Farther it has been obſerved by Sir *Iſaac Newton* that the rays of light in their paſſage from one ſuperficies of a refracting body to the other, undergo alternate fits of eaſy tranſmiſſion and reflection, ſucceeding each other at equal intervals: inſomuch that if they reach the farther ſurface in one ſort of thoſe fits, they ſhall be tranſmitted; if in the other kind of them they ſhall rather be reflected back. Whence the rays that proceed from *B* to *C*, and emerge in the line *CD*, being in a fit of eaſy tranſmiſſion, the ſcattered rays that fall at a ſmall diſtance without theſe on either ſide (ſuppoſe the rays that paſs along the lines *BE*, *BG*) ſhall fall on the ſurface in a fit of eaſy reflection and ſhall not emerge; but the ſcattered rays, which paſs at ſome diſtance without theſe laſt, ſhall arrive at the ſurface of the globule in a fit of eaſy tranſmiſſion and break through that ſurface. Suppoſe theſe rays to paſs in the lines *BF*, *BH*, the former of which ſhall have had one fit more of eaſy tranſmiſſion and the latter one fit leſs, than the rays that paſs from *B* to *C*. Now both theſe rays when they go out of the globule, will proceed by the refraction of the water in the lines *FI*, *HK*, that will be inclined almoſt equally to the rays incident upon the globule, which come from the ſun: but the angles of their inclination will be leſs than the angle in which the rays emerging in the line *CD* are inclined to thoſe incident rays. And after the ſame manner rays ſcattered from the point *B* at a certain diſtance without theſe will emerge out of the globule, while the intermediate rays are intercepted; and theſe emergent rays will be inclined to the rays incident upon the globule in angles ſtill leſs

Opt. Book. III Part 4. at the beginning.

Fig. 477.

Opt. Book II. Part. 3. prop. 12.

leſs than the angles, in which the rays *FI* and *HK* are inclined to them; and without theſe rays will emerge other rays that ſhall be inclined to the incident rays in angles ſtill leſs. Now by this means may be formed of every kind of rays, beſides the principal arch, which goes to the formation of the rain-bow, other arches within every one of the principal of the ſame colour, though much more faint; and this for diverſe ſucceſſions, as long as theſe weak lights, which in every arch grow more and more obſcure, ſhall continue viſible. Now as the arches produced by each colour will be variouſly mixed together, the diverſity of colours obſerved in theſe ſecondary arches may very poſſibly ariſe from them. In the darker colours theſe arches may reach below the bow and be ſeen diſtinct: in the brighter colours theſe arches are loſt in the inferior part of the principal light of the rain-bow; but in all probability they contribute to the red tincture which the purple of the rain-bow uſually has, and is moſt remarkable when theſe ſecondary colours appear ſtrongeſt. However theſe ſecondary arches in the brighteſt colours, may poſſibly extend with a very faint light below the bow and tinge the purple of the ſecondary arches with a rediſh hue. The preciſe diſtances between the principal arch and theſe fainter arches depend on the magnitude of the drops, wherein they are formed. To make them any degree ſeparate it is neceſſary the drop be exceeding ſmall. It is moſt likely they are formed in the vapour of the cloud, which the air, being put into motion by the fall of the rain, may carry down along with the larger drops: and this may be the reaſon why theſe colours appear under the upper part of the bow only, this vapour not deſcending very low. As a farther confirmation of this, theſe colours are ſeen ſtrongeſt when the rain falls from very black clouds, which cauſe the fierceſt rains by the fall whereof the air will be moſt agitated. So far Dr. *Pemberton*.

To the like alternate fits of eaſy tranſmiſſion and reflection in the paſſage of light through the leaſt globules of water, Sir *Iſaac Newton* aſcribes thoſe little rings of colours which ſometimes appear round about the ſun and moon. Opt. p. 288. obſ. 13.

LEMMA

508. The tangent of the ſum of two angles, is to the ſum of their tangents, as the ſquare of the radius, to the ſquare of the radius diminiſhed by the rectangle under the tangents: and the tangent of the difference of two angles, is to the difference of their tangents, as the ſquare of the radius, to the ſquare of the radius increaſed by the rectangle under the tangents.

Fig. 478, 479. Let *RA* and *RB* be tangents of two angles *ROA*, *ROB*. Then as *AB*, the ſum or difference of the tangents, is to *AO*, the ſecant of either of the angles, ſo let *AO* be to *AC*, to be placed from *A* towards *B*. Again as *RC* is to RO, ſo let RO be to RD; and RD will be the tangent of the ſum or difference of the two angles ROA, ROB. For joining CO, by the firſt of theſe proportions the triangles AOB, ACO will be equiangular[a]; and ſo the angle AOB is equal to ACO, or to ROD by the ſecond proportion[b]. Hence in fig. 478, becauſe $AC = \frac{AOq}{AB} = \frac{RAq + ROq}{RB + RA}$, we have $RC = (AC - AR =) \frac{RAq + ROq}{RB + RA} - RA = \frac{ROq - RB \times RA}{RB + RA}$; whence $RD = \frac{RB + RA}{ROq - RB \times RA} \times ROq$. By a like proceſs fig. 479. we have $AC = \frac{RAq + ROq}{RB - RA}$; whence $RD = \frac{RB - RA}{ROq + RB \times RA} \times ROq$. Q. E. D.

[a] Euc. VI. 6.

[b] Euc. VI. 8.

509. *Corol.* 1. Hence the tangent of the ſum of any number of given angles, or the tangent of any multiple of a given angle, may be eaſily computed. Put $RO = r$, $RA = a$, $RB = b$, then the tangent of the ſum of the angles whoſe tangents are a and b, that is $RD = \frac{b + a}{rr - ab} \times rr$; call this tangent x; then for the ſame reaſon, the tangent of the ſum of this laſt angle and of a third angle, whoſe tangent is c, is $\frac{x + c}{rr - xc} \times rr$ or (by ſubſtituting the value of x) $\frac{rr \times \overline{a + b + c} - abc}{rr - ab - ac - bc}$, the tangent of the ſum of three angles whoſe tangents are a, b, c; and ſo on.

510. *Corol.* 2. Now put $a = b = c$; and for the tangent of a double angle we have $\frac{2a}{rr - aa} rr$; and for the tangent of a treble angle $\frac{3arr - a^3}{rr - 3aa}$; and ſo on.

PROPOSITION V.

511. *The apparent ſemidiameter of any rain-bow, or the greateſt angle under an incident and the emergent ray after any given number of ſucceſſive reflections, being given; to find the ratio of refraction.*

Let m be the given number of ſucceſſive reflections increaſed by an unite, and ſuppoſing the angles ABC, ABD to be the angles of incidence and refraction ſought, let the angle $ABE = m \times ABD$, and the angle CBE, or $m \times ABD - ABC$, will be half the given angle under the incident and the emergent ray after $m - 1$ reflections[c]. Put the common radius $AB = r$, the unknown tangent of refraction $AD = a$, and the tangent of incidence $AC = ma$*, alſo $AE = x$, and t for the tangent of the given angle CBE anſwering to the radius r. Then by the lemma $t : x - ma :: rr : rr + xma$; whence $t = \frac{x - ma}{rr + xma} rr$.

Fig. 480.

[c] Art. 488.

* Art. 491.

Caſe 1. In the firſt rain-bow $m=2$; whence $t=\frac{x-2a}{rr+2xa}rr$, and by art. 520, $x=\frac{2a}{rr-aa}rr$ the tangent of $2ABD$. Subſtitute this value for x in the former equation and by reduction it becomes $a^3-\frac{3}{2}taa-\frac{1}{2}trr=0$. By reſolving this equation the tangent a of the angle of refraction will be given, and the tangent of the angle of incidence $AC=2a$ by art. 491. whence the ratio of their ſines is given by the tables.

Caſe 2. In the ſecond rain-bow $m=3$, whence $t=\frac{x-3a}{rr+3xa}rr$, and by art. 510, $x=\frac{3arr-a^3}{rr-3aa}$, the tangent of $3ABD$. Subſtitute this value for x and you will find $a^4+\frac{8}{3}\frac{rr}{t}a^3-2rraa*-\frac{1}{3}r^4=0$; or putting $T=\frac{rr}{t}$ the tangent of half the angle of this bow[a], $a^4+\frac{8}{3}Ta^3-2rraa*-\frac{1}{3}r^4=0$. The ſame method ſerves for other bows to infinity. [a] Art. 488.

512. *Corol.* In the firſt caſe putting T for $2a$ or AC the tangent of the angle of incidence, and ſubſtituting $\frac{1}{2}T$ for a in the former equation $a^3-\frac{3}{2}taa-\frac{1}{2}trr=0$, it is changed to this $T^3-3tTT-4rrt=0$, the ſame as Dr. *Halley*'s, who propoſed this problem as an expeditious method for finding the ratio of refraction in any fluid, by obſerving (when the ſun is low and ſhines very bright) the angle under an incident and the emergent ray from a drop of any fluid hanging at the end of a capillary tube. See his examples Phil. Tranſ. N°. 267. and alſo the Rev^d. Dr. *Morgan*'s Diſſertation upon the Rain-bow among the Notes upon *Rohault*'s Phyſicks. P. 3. Ch. 17.

Having done with the rain-bows I proceed now to conſider the rays which emerge from a drop after two refractions without any reflection, in order to account for halo's.

PROPOSITION VI.

Fig. 481. 513. *When a ray of light* SABL *is refracted through a ſphere at* A *and* B, *without any intermediate reflection, the angle* LMN, *under the incident ray* SAMN *and the emergent ray* LBM *produced, is equal to twice the exceſs of the angle of incidence* OAM *above the angle of refraction* OAB: *and by conſequence increaſes perpetually while the angle of incidence increaſes.*

For the external angle LMN is equal to the ſum of the internal angles MAB, MBA; which are equal to each other, becauſe of the equal refractions at A and B; as appears by conceiving a ray to go out both ways along the chord AB. And one of theſe angles equals the exceſs of OAM above OAB: and ſince the increments of OAM are always bigger than thoſe of OAB *, the exceſſes of the larger increments above the ſmaller will continually augment the angle BAM, and conſequently the whole angle LMN. *Q. E. D.* * Art. 421.

514. *Corol.* 1. Draw a diameter $POQE$ parallel to the incident ray SAM; and let it cut the emergent ray BL in E; and while the angle LMN increaſes from nothing, the line ME will decreaſe perpetually. For let the parallelogram $EMAH$ be compleated, and let AB and PQ be produced till they meet in T; and the equiangular triangles AMB, BET, AHT will be equicrural. Now if ME or AH be ſuppoſed to increaſe or only to keep the ſame while the oppoſite angle AOH decreaſes, the adjoining ſide HO muſt increaſe, and conſequently the ſum of AH and HO or of HT and HO that is OT muſt alſo increaſe. But by art. 476. the line OT decreaſes perpetually; therefore the ſuppoſition we made that AH or ME increaſed or kept the ſame is falſe; and by conſequence it decreaſes perpetually while the angle of incidence increaſes; or while the angle LMN increaſes[a]. Fig. 482.

[a] Art. 513.

515. *Corol.* 2. Hence while the angle LMN increaſes, the line OE decreaſes perpetually. For it is eaſy to ſhew that ME and OE are equal.

PROPOSITION VII.

516. ***When parallel rays fall upon the ſurface of a ſphere and emerge from it after two refractions without an intermediate reflection, their denſity at the eye of a ſpectator placed at a great diſtance from the ſphere, will decreaſe perpetually, while the angles under the incident and the emergent rays increaſe.***

For ſuppoſing the ſame lines as before, let $SabF\lambda$ be the neareſt ray to $SABFL$, and let them croſs each other in F, and then fall perpendicular upon $L\lambda$ ſuppoſed to be the diameter of the pupil of the eye. From the center O draw OHh perpendicular to the incident rays produced, OIi perpendicular to the refracted rays AB, ab; and OKk perpendicular to the emergent rays; and alſo LR, MS, AT perpendicular to the diameter PQ, drawn parallel to the incident rays; and let TA produced cut Sa in α. Then ſince OI is to OH and to OK as the leſſer of the ſines of incidence and refraction to the bigger; and alſo Oi to Oh and to Ok in the ſame ratio; disjointly we have Ii to Hh and to Kk in the ſame given ratio; and conſequently Hh equal to Kk. But the triangles KFk, $LF\lambda$ are equiangular, and alſo MES and LER. Therefore $A\alpha$ or Hh or $Kk : L\lambda :: FK : FL$, and AT or $MS : LR :: EM : EL$; and by conſequence $A\alpha \times AT : L\lambda \times LR :: FK \times EM : FL \times EL$. Now ſuppoſing the whole figure to be turned about the axis PQ; ſince the ſame rays that paſs through the ring deſcribed by $A\alpha$ will alſo paſs through the ring deſcribed by $L\lambda$; ſuppoſing none of them to be ſtopt in the ſphere, it follows that their denſity at L is to their denſity at A reciprocally as the rings; that is directly as $A\alpha \times AT$ to $L\lambda \times LR$, or as $FK \times EM$ to $FL \times EL$ (becauſe the peripheries deſcribed by A and L are as their ſemidiameters AT, LR.) And the denſity of the incident rays being every where the ſame, the denſity Fig. 483.

of the emergent rays at L is as $FK\times EM$ directly and as $FL\times EL$ inverſely; and by conſequence directly as $FK\times EM$, when the eye at L is remote; becauſe the points E, F never go far from the ſphere[a]. Now while the angle LMN increaſes, the line FK decreaſes perpetually[b], and ſo does EM[c], and therefore the denſity of the rays falling upon a remote eye at L perpetually decreaſes. *Q. E. D.*

[a] Art. 432. 515.

[b] Art. 420. 432.

[c] Art. 514.

This propoſition may alſo be demonſtrated in diſtinct parts as follows.

517. 1ſt. The denſity of the rays which fall perpendicular upon the line $L\lambda$ in any one plane of emergence, is as FK directly and FL inverſely; and when L is remote, directly as FK. For ſuppoſing all the incident rays to be tranſmitted from $A\alpha$ to $L\lambda$, their denſity in $L\lambda$ will be to their denſity in $A\alpha$ as ($A\alpha$ or Hb or Kk to $L\lambda$, or as) FK to FL: and the line FL will be invariable when L is remote, becauſe the focus F never goes far from the ſphere[d].

[d] Art. 432.

518. 2ly. The denſity of the rays which fall upon the periphery deſcribed by the point L, turned about the axis PQ, is as EM directly and EL inverſely: and when L is remote, directly as EM. For ſuppoſing all the incident rays to be tranſmitted, their denſity in the periphery deſcribed by the point L, will be to their denſity in the periphery deſcribed by the point A, reciprocally as theſe peripheries, that is directly as the ſemidiameter AT or MS, to the ſemidiameter LR, that is as EM to EL: and the line EL will be invariable when L is remote, becauſe the point E never goes far from the ſphere[e].

[e] Art. 515.

519. 3ly. It is obſervable that as F is the focus of thoſe rays which emerge from the ſphere in any one plane of incidence and emergence, ſo E is the focus of thoſe that emerge in a conical ſurface deſcribed by the revolution of EL about the axis OE, to which ſurface all the planes of emergence are perpendicular.

520. 4ly. When a ſolid pencil of rays falls upon a plane ſurface and are uniformly denſe in every part of it; we may conſider the incident rays as conſiſting of innumerable phyſical planes of rays, which fall upon the ſurface in as many phyſical lines, parallel to one another. And then if their denſity in any one of theſe lines be given, their denſity upon the ſurface will be as the denſity of the lines; and if the denſity of the lines be given, the denſity of the rays upon the ſurface will be as their denſity in any one line upon it; and conſequently if neither the one nor the other be given, the denſity of the rays upon the ſurface will be as their denſity in any one line and as the denſity of the lines conjointly.

521. 5ly. To apply this rule to the denſity of the rays upon the circular ring deſcribed by the revolution of $L\lambda$; ſince its breadth $L\lambda$ is ſuppoſed very ſmall in compariſon to its diameter, the planes of emergence will divide any ſmall part of it into lines, as $L\lambda$, nearly parallel to one another. And by conſequence the denſity of the rays in this little part, will be as

their denſity in any one line $L\lambda$ and as the denſity of the lines $L\lambda$ conjointly; that is as $\frac{FK}{FL} \times \frac{EM}{EL}$ by the 1ſt and 2d of theſe rules; and when L is remote, as the rectangle $FK \times EM$, as before.

522. 6ly. Now let heterogeneal rays be refracted through the ſphere; and ſince the whole quantity of refraction increaſes gradually as the rays go farther and farther from its center[a], theſe heterogeneal rays will therefore be gradually more and more ſeparated from one another[b]; and upon this account will fall thinner upon the eye at $L\lambda$ than if they were all homogeneal.

523. 7ly. We have all along ſuppoſed that all the incident rays are tranſmitted through the ſphere. But it is certain that ſome of them are reflected both at the firſt and ſecond ſurface, as in the caſe of the rain-bows. And it is a common obſervation, that as rays fall more obliquely upon any ſurface, the more of them will be reflected; and therefore the rays that go farther and farther from the center of the drop, and conſequently fall more and more obliquely upon its ſurfaces, will be more and more diminiſhed by reflections; and ſo the emergent light will decreaſe ſtill faſter, or in a greater proportion than of the decreaſe of the rectangle $FK \times EM$.

524. 8ly. If the ſun be viewed through ſo large a ſphere as to ſubtend an angle at the eye as large or larger than the ſun it ſelf would ſubtend at the naked eye; his body will appear biggeſt through the ſphere when it is held in a line drawn directly from the eye to the ſun; and will appear to decreaſe gradually while the ſphere is moved ſideways from that line. This ſhall be demonſtrated in another place, and may be eaſily tryed by looking at a lighted candle or any bright object through a glaſs full of water. In this caſe the apparent brightneſs of the ſun, or the real brightneſs of his picture upon the retina, would vary directly as the denſity of the rays in any one pencil when they fall on the eye, and inverſely as the magnitude of the picture: becauſe if the magnitude of the picture was invariable, its brightneſs would be as the denſity of the rays in any one pencil; and if the denſity was invariable, its brightneſs would be inverſely as its magnitude. But ſince the angle ſubtended at the eye by a globule in a cloud has no ſenſible magnitude, the picture of the ſun upon the retina, formed by rays that come through it, can have no variety of magnitudes; being always but a point which affects the ſenſe not by its magnitude but by its brightneſs only. Therefore while this globule is moved ſideways the brightneſs of the ſun, ſeen through it, varies directly as the denſity of the rays at the eye which came from a ſingle point of the ſun; becauſe the denſity of the rays that come at once from all the points of the ſun varies in the ſame proportion as their denſity in any one pencil varies.

525. 9ly. Therefore when the ſun ſhines upon a large cloud of ſuch globules, the light will appear ſtrongeſt in thoſe globules which lye directly

between

[a] Art. 513.

[b] Art. 171.

between the eye and the ſun, and will decay more and more in the globules that lye farther and farther from that line or from the apparent place of the ſun. In this concluſion I agree with *Hugenius* in his diſſertation concerning Corona's and Parhelia, printed in the next chapter; though I differ from him in the reaſon he gives for it, namely becauſe the drops that lye neareſt to the ſun make the largeſt images of him[a]; (maximam ſolis imaginem exhibent); whereas theſe images being leſs than the globules have no ſenſible apparent magnitude; and therefore appear brighter or duller only becauſe the rays fall thicker or thinner upon the eye.

526. Nevertheleſs to think different from ſo great a maſter of reaſon as Sir *Iſaac Newton*, is but little better than a degree of uncertainty; or at leaſt a ſort of pain which the mind would willingly be freed from. I have therefore been very particular in demonſtrating this concluſion becauſe I cannot reconcile it with one of his[b]. Where he ſays that the light which comes through drops of rain by two refractions without any reflection ought to appear ſtrongeſt at the diſtance of about 26 degrees from the ſun; [that is when the angle *LMN* is about 26 degrees;] and to decay gradually both ways as the diſtance from him increaſes and decreaſes. And that the ſame is to be underſtood of light tranſmitted through ſpherical hail-ſtones. And if the hail be a little flatted, as it often is, the light tranſmitted may grow ſo ſtrong at a leſs diſtance than that of 26 degrees as to form a halo about the ſun or moon. I have already obſerved that *Hugenius* was of a contrary opinion; and beſides he tells us in another place of his diſſertation that though he had examined all the reflections and refractions of the ſun's rays in globules of water, yet he could find no cauſe at all for the appearance of a ring of light about him of the uſual bigneſs of a halo[c]. And thence he began to ſuſpect that they are formed by hail-ſtones that have opake globules of ſnow in their centers. In this opinion Sir *Iſaac Newton* ſeems to agree with him. For he goes on in theſe words[d],——which halo, as often as the hail-ſtones are duly figured, may be coloured; and then it muſt be red within by the leaſt refrangible rays, and blue without by the moſt refrangible ones; eſpecially if the hail-ſtones have opake globules of ſnow in their centers to intercept the light within the halo (as *Hugenius* has obſerved) and make the inſide thereof more diſtinctly defined than it would otherwiſe be. For ſuch hail-ſtones, though ſpherical, by terminating the light by the ſnow, may make a halo red within and colourleſs without, and darker within the red than without as halo's uſe to be. For of thoſe rays which paſs cloſe by the ſnow, the rubiform will be leaſt refracted, and ſo come to the eye in the directeſt lines. The light which paſſes through a drop of water after two refractions and three or more reflections, is ſcarce ſtrong enough to cauſe a ſenſible bow [as appears by the different ſtrength of the firſt and ſecond rain-bows] but in thoſe cylinders of ice by which *Hugenius* explains the parhelia it may perhaps be ſenſible. This

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[a] Art. 530.

[b] Opt. p. 155.

Fig. 481.

[c] Art. 528.

[d] Opt. p. 155.

is all I can find of Sir *Iſaac Newton*'s concerning this ſubject, excepting another ſhort paſſage where he ſeems to ſuggeſt a reaſon why halo's appear oval, an inſtance of which he had mentioned juſt before; and then adds theſe words. By its being oval, and remoter from the moon below than above, I conclude that it was made by refraction in ſome ſort of hail or ſnow floating in an horizontal poſture, the refracting angle being about 58 or 60 degrees[a]. In this alſo I muſt confeſs I cannot reſt ſatisfied; because all halo's, though formed by ſpherical hail with ſpherical globules of ſnow in their centers, muſt appear oval in the poſition he mentions upon account of the flattiſh figure of the ſky, as it is explained in the 167th article.

[a] Opt. p. 291.

CHAPTER XI.

Concerning Corona's and Parhelia, commonly called Halo's and Mock-Suns.

IN this chapter I have tranſlated *Hugenius*'s whole diſſertation upon this curious ſubject; and as this excellent Author intended to demonſtrate the more difficult things at the end of all, which the learned Editors of his Poſthumous Works have ſupplyed in few words, I have done the ſame more at large in an Appendix to this chapter; which contains alſo the obſervations referred to in this diſſertation. Deſign.

I. Corona's what and how large.

527. Though the cauſes of corona's and parhelia, which ſhall here be aſſigned are much alike, yet I will firſt treat of corona's, becauſe they are eaſier to be explained; and being underſtood, will facilitate the explanation of parhelia. Now corona's are circular rings of light, which by day appear round about the ſun, and ſometimes by night about the moon; ſometimes white, and ſometimes, when brighteſt, adorned with various colours like the rain-bow. Their apparent diameter is generally about 44 or 45 degrees, but ſometimes a larger ſort have appeared, about 90 degrees in diameter or more. It is ſaid that a great many ſuch rings have appeared together, with the ſun in their centers. I my ſelf have often obſerved thoſe rings of 45 degrees; the firſt of which, being the moſt lively, gave me occaſion to take notice of the moſt remarkable appearances in it, in order to attempt their explication. I obſerved this ring was made up of various colours, but fainter than thoſe in the rain-bow; that its inward edge was red and its outward a pale blue, much inclining to white; that the ſpace within the ring (called the area) was darker than that on the outſide of it, which appeared every way pretty clear, excepting that it was covered with ſome ſmall pellucid and whitiſh clouds. I found the diameter of the ring about 45 degrees, by a groſs manner of meaſuring it, which was by holding my cane at arms length and obſerving what part

of it covered the diameter of the area; and thence I computed the angle as near as I could. This obſervation was made 30. Mar. 1652. I remember after this obſervation I looked into what *Des Cartes* had wrote concerning the cauſes of corona's. He would have them to be generated by the rays of the ſun refracted through flat ſtars of pellucid ice: which I did not like, becauſe it follows from thence, as he himſelf confeſſes, that the ſpace within the crown ſhould appear brighter than that without, contrary to what I had obſerved.

II. Whence cauſed.

528. I therefore thought it proper to ſearch after ſome other cauſe of this appearance, though *Des Cartes* aſſerts, *that no other can be found in the heavens to produce any thing like it*. I examined all the refractions and reflections of the ſun's rays through globules of water, but found nothing that could cauſe a circle of that bigneſs. I therefore framed to my ſelf other figures conſiſting of congealed drops, but in vain; ſo hard it is to form an idea of a thing never ſeen. But being drawn into the ſame thoughts by the appearance of 5 ſuns at *Warſaw* in the year 1658; and being diligently attentive to the figure of the corona deſcribed above, at laſt I hit upon their true cauſe; and a little after upon that of the parhelia. For I make no ſcruple to call thoſe cauſes true, whoſe effects agree ſo exactly with obſervations, as to make it ſeem unneceſſary to ſearch for others. Conſidering therefore that certain particles flying about among the clouds afforded materials for theſe meteors; as appeared ſufficiently from hence, that the corona kept in the ſame place though the clouds were diſperſed; conſidering alſo that the dulneſs of the ſky within the corona was an argument that the particles within that circle did not tranſmit the ſun's light ſo copiouſly as when they flew out of it; I ſoon apprehended that the corona might be formed, if each of theſe particles were round globules conſiſting of a coat or ſhell of tranſparent ice or water, that contained an opake kernel within it. For I eaſily perceived, if a great number of ſuch particles were diſperſed between us and the ſun, that thoſe which lay within a certain diſtance from the ſun, could tranſmit no rays to the eye, whereas others more remote would let them paſs; as will appear preſently. Now it is not only probable but matter of fact, that ſuch ſort of hail-ſtones are diſperſed among the clouds; becauſe as *Des Cartes* atteſts, they ſometimes fall down upon the ground. For ſpeaking in his Meteors about the formation of ſuch particles, he ſays, *hence it comes to paſs, that as the outward ſhell of ſuch particles uſually conſiſt of a continued pellucid ice, there is often found ſome ſnow at their centers, as appears by breaking them*. Nor will it appear improbable that particles of ſnow ſhould ſtick in the middle of globules of water, conſidering that theſe globules are ſuſtained by exhalations or vapours raiſed upwards; by which means the water is hindered from deſcending and from deſerting the ſnow in its center. Neither is there any reaſon, why the kernels which are of the ſame

gravity

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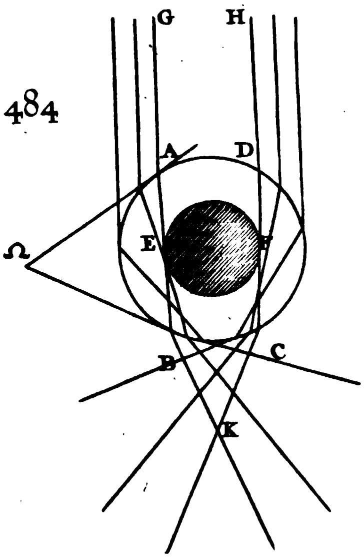

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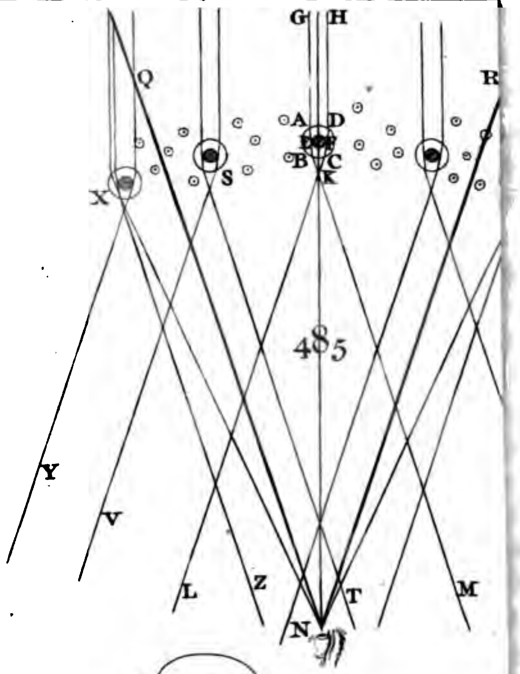

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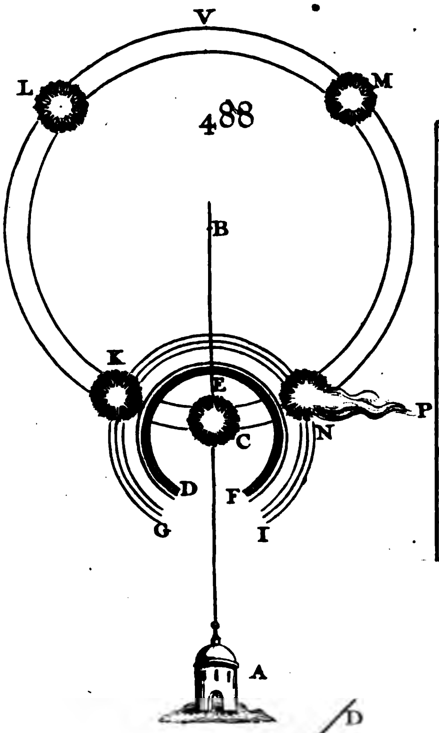

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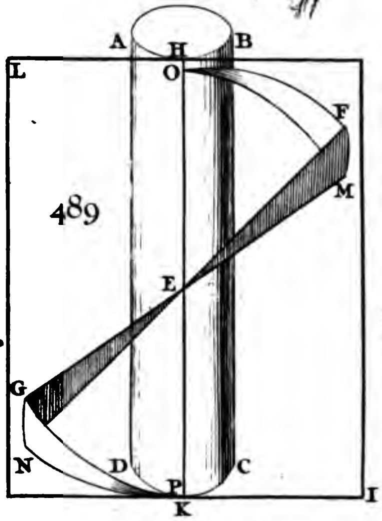

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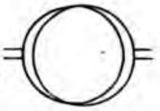

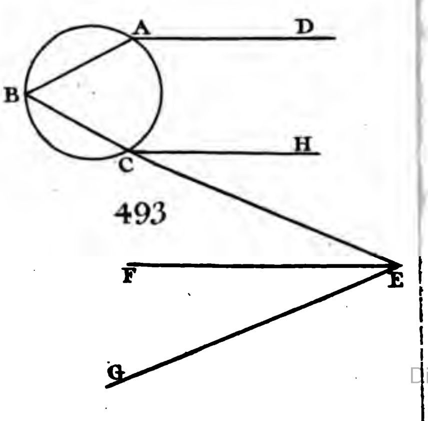

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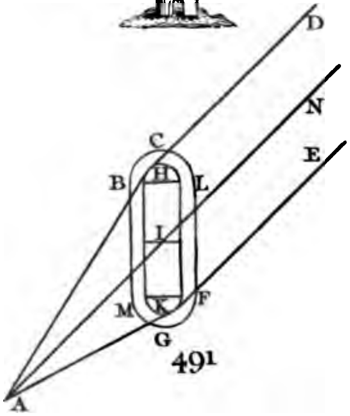

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gravity as the water, ſhould recede from it either by aſcending or deſcending; at leaſt thoſe hail-ſtones which deſcend to the earth do plainly prove the matter to be fact. Now allowing them to be ſuſtained by aſcending exhalations, it ſeems neceſſary that they ſhould be very ſmall, perhaps not ſo big as grains of turnepſeed; nevertheleſs this ſmalneſs does not hinder the perfection of their figures, nor the juſt proportion between the bulk of the whole and of the kernel within, which I ſhall inquire into by and by. Having made it appear that ſuch mixed corpuſcles are formed in the air, I ſhall ſhew afterwards how all the appearances of corona's are produced by them.

III.
In what manner.
Fig. 484.

529. The better to underſtand this matter, let us conſider one of theſe globules under a larger form, in order to apprehend how the rays of the ſun are refracted through it. Let *ABCD* repreſent a globule, with a kernel of ſnow *EF* in the middle of it, and let us ſuppoſe the rays coming from *GH* to fall upon the ſide *AD*. It is manifeſt they will ſo be refracted at the ſurface *AD* as to bend inwards; whence it follows that a great number of them muſt ſtrike upon the kernel *EF*. Let *GA* and *HD* be the rays which after refraction touch the ſides of the kernel *EF*, and let them be refracted again at *B* and *C*, and emerge in the lines *BK*, *CK* croſſing each other in the point *K*; whoſe neareſt diſtance from the globule is ſomewhat leſs than its ſemidiameter[a]. Therefore if *BK* and *CK* be produced towards *M* and *L*, it follows that no light coming from the ſun through the globule, can proceed to the eye any where placed within the angle *LKM*, or rather within the cone repreſented by *LKM*. Becauſe all the rays which paſs by the kernel, will croſs each other under a greater angle than *LKM*, and at a point nearer to the globule than the point *K*[*], and therefore will paſs by the eye placed any where within the angle *LKM*, but will fall upon it when placed out of that angle. What is ſaid of this globule, belongs to all others of the ſame ſhape. That is to ſay every one of them carries a ſhady cone behind it, in which the eye being placed can perceive no light coming through the globule, but being removed out of it, will immediately perceive the light of the ſun. Let the eye be at *N*, and conceive a cone whoſe vertex is *N* and whoſe ſides *NR*, *NQ* are parallel to the ſides *KL*, *KM* of the former cone. It is certain that none of the globules which include equal kernels of ſnow to that in the globule *ABCD*, and which are placed any where within the cone *QNR*, can direct the rays of the ſun towards the eye at *N*. For if from any ſuch globule as *S*, two lines *SV* and *ST* be drawn parallel to *KL* and *KM*, repreſenting in this figure the ſides of a ſhady cone behind *S*, it appears that the eye at *N*, in the vertex of the cone *QNR*, falls within the cone *VST*, and conſequently receives no rays coming through the globule *S*. The ſame may be ſaid of all the globules within the cone *QNR*, becauſe

Fig. 485.

Fig. 485.

[a] Art. 61, 65.

[*] Art. 512. 513.

the eye at *N* is placed within the ſhady cone belonging to every one of them, as appears by the figure.

IV. The dark area with light about it, proved.

530. But if any other globule, as *X*, lying out of the cone *QNR*, be taken into conſideration, it appears that the eye at *N* lyes out of its ſhady cone *YXZ*; and conſequently thoſe rays of the ſun, which are more refracted through the globule *X* than the ray *XZ*, will come to the eye; ſo that this globule and all others out of the cone *QNR* will appear enlightened; whereas thoſe within it will appear obſcure. It is plain therefore that a certain area or ſpace quite round the ſun ought to appear dark or obſcure; and that the ſpace round about this area muſt appear more luminous; and more ſo in the parts that are neareſt to the obſcure area. Becauſe it might eaſily be demonſtrated that the globules that are neareſt to the cone *QNR* exhibit the largeſt image of the ſun[a]. It is plain alſo that a corona ought to be produced in the ſame manner, whatever be the ſun's altitude, by reaſon of the ſpherical figure both of the globules and of the kernels within them. Whoever has a mind to make a proper experiment to repreſent this appearance, let him expoſe to the ſun a thin glaſs bubble filled with water, that has an opake ſphere in the middle of it, which may eaſily be contrived; and he will find, that he cannot ſee the ſun's image in it, unleſs he removes it to a certain diſtance from the line drawn from his eye to the ſun: and that as ſoon as he perceives the light, the image of the ſun will immediately appear the brighteſt, and coloured red, for the ſame reaſon as in a glaſs priſm or rather a water priſm, if ſuch a one could be made. For the ray *GA*, for example, ſuffers the ſame refractions, at its ingreſs into the globule at *A* and at its egreſs at *B*, as it would do in paſſing through a priſm, whoſe ſides *A*Ω, *B*Ω touch the globule at *A* and *B*. And the kernel *EF*, which terminates the light on one ſide, conduces much to the appearance of the colours of the rays *AB* which paſs by it. But I have no deſign to inquire into the cauſes why theſe colours appear through priſms. I confeſs I am entirely ignorant of their production, nor do I believe that any one can diſcover it till we receive ſome greater light into the knowledge of nature. All that I would obſerve is that the red colour which appears in the glaſs bubble ought alſo to be ſeen in the globules that are neareſt to the outſide of the cone *QNR*; which growing bigger in the remoter globules produces other colours as in the rain-bow. I judge the breadth of the corona, equal to the apparent ſpace through which the colours are ſpread when the drops are moſt enlightened; although in reality the breadth is only terminated by the ſnow on the inſide next to the ſun and not on the outſide; juſt as in the rain-bow, which alſo, as *Des Cartes* has explained it, is only terminated on the red ſide and not on the other. Theſe corona's do alſo often appear about the moon for the ſame reaſons as about the ſun: but the colours are generally ſo weak as to appear only white: theſe white corona's I have often ſeen about the ſun,

Fig. 484.

[a] See Art. 525.

ſun, when the ſpace within them appeared ſcarce darker than that without. Now this happens when there is leſs plenty of ſuch globules. For the more there are the more lively the colours appear, as is manifeſt from obſervations; by which it appears, that when the area within the corona is darker, that is when the globules are more numerous, the colours appear more vivid. But the colours within the corona appear more lively when parhelia and paraſelenæ, that is, mock-ſuns and mock-moons appear with them, for another reaſon to be given when I treat of theſe apappearances.

V.
The diameter of a corona conſidered.
Fig. 485.
Fig. 486.

531. Let us now conſider the apparent diameter of the corona, which, as we ſaid before, is generally about 45 degrees. This depends upon the magnitude of the dark kernel *EF*. For the greater it is in compariſon to the whole globule *ABCD*, ſo much the greater is the angle *BKC*, to which the angle of the cone *QNR* is equal and meaſures the apparent diameter of the corona, as we have ſhewn above. Beſides the rays *GABK*, *HDCK*, paſſing through the globule *ABCD*, and touching the kernel in *E* and *F*, draw from the center *M* the line *MKO*, paſſing through *K* the interſection of the rays, and meeting the line *DC*, produced in *O*: let the lines *HD* and *KC* produced meet in *L* within the globule Now in the triangle *DLC*, the angles *FDL*, *FCL* are equal, becauſe the ray *HD* is refracted at *D*, in entering into the globule, juſt as much as at *C*, in going out of it. But the angle *FDL* is equal to *DOK*, becauſe *HL* and *DO* are parallel; therefore in the triangle *KOC*, the angles *KOC*, *KCO* are alſo equal; and both together are equal to the angle *CKM*, which is half of *CKB*. Now ſince *CKB* is obſerved to be 45 degrees, its half *MKC* is $22\frac{1}{2}$; and the half of this, that is *KOC* is $11\frac{1}{4}$. Therefore in the triangle *OMF*, right angled at *F*, the ratio of the ſides is given, and taking the radius *OM* to conſiſt of 100000 parts, *MF* the ſine of $11\frac{1}{4}$ degrees will be 19509 and *OF* 98078 ſuch parts. The ratio of *OM* to *OD* is the ſame as of the ſine of the angle of refraction *MDO* to the ſine of the angle of incidence *MDL* or *DMN* * that is as 187 to 250. Therefore ſince *OM* contains 100000 parts, *OD* will contain 133690, from whence taking *OF* 98078, there remains *FD* 35612; but *MF* was 19509. Therefore the proportion of the ſides of the right angled triangle *MFD* being known, the ratio of either of them to the baſe *MD* will be given, which *MD* will be found 40605. Therefore *MD* the ſemidiameter of the globule, muſt be to *MF* the ſemidiameter of the kernel of ſnow, as 40605 to 19509 or as 1000 to 480 very nearly, to make the diameter of the corona 45 degrees. By the ſame method, to make the diameter of a corona 90 degrees, we ſhall find the ratio *MD* to *MF* muſt be as 1000 to 680; and as 1000 to 473 if the diameter be 44 degrees.

* Art. 221.

VI.
The formation of the globules conſidered.

532. Concerning the formation of this ſort of globules, it is probable that at firſt they were globules of ſoft ſnow, which are rounded by a con-

tinual agitation in the air, and thawed on their outſides by the heat of the ſun. The appearance of corona's requires an exact temperature of heat and cold in the air, that the globules of ſnow may be gradually thawed a little, and be checked again by a ſmall increaſe of cold. But when they are thawed about half way through their diameters, or a little more, there is a cone of rays reflected from their backſides inwards, whoſe heat will prevent their being congealed again, though a greater degree of cold ſhould follow. Now by the warmth of this cone which is greateſt at its apex, the ſnow in the center will be rounded and poliſhed, becauſe the globule is continually turning different ſides to the ſun, or rather becauſe the warmth of the cone ſpreads round the whole globule. And this roundneſs is neceſſary to the production of corona's. It is probable indeed that a great many globules may be unequally thawed, but if their kernels be not rounded they cannot produce any joint effect. And perhaps by this external warmth, the cold is driven into the middle of the globule (as we ſee it often happens in winter time) and prevents the farther diſſolution of the kernel of ſnow. I ſuppoſe the outſides of theſe globules to be watery, as is moſt probable, becauſe the ſurface of water is generally ſmoother and fitter to cauſe regular refractions than if it was congealed. Yet ſometimes it may happen that the globules, ſo formed as deſcribed above, may be congealed again, and may ſtill continue round and tranſparent enough to produce a corona. Now beſides this collection or cone of rays, which melts the ſnow half way through, there is alſo another collection, which may form and preſerve other kernels of ſnow that are larger in proportion to the whole globule; and which may produce another corona as broad again as the former; for ſuch corona's have ſometimes been obſerved as will appear afterwards. It is alſo to be obſerved that this collection of rays ſeems to be the reaſon why corona's are generally obſerved to be of ſuch determinate diameters; nevertheleſs it is not impoſſible that corona's may appear of any diameters; ſince nothing hinders the globules in a temperate air, from melting very ſlowly, and after a certain part of them is diſſolved, from keeping that proportion ſo long, that corona's may appear for hours together without ſenſible change. Hence if a certain collection of theſe globules, diſſolved to a certain degree, and another collection of them, diſſolved to another degree, ſhall happen to be ranged in a higher or lower region than the former, and other collections diſſolved to other degrees be placed in other regions; there will appear ſeveral corona's of different diameters at the ſame time; and accordingly ſome authors tell us they have ſeen ſix ſuch parallel corona's concentrick to the ſun [a].

Fig. 487.

VII. Concerning parhelia and paraſelenæ.

533. Having hitherto conſidered the cauſes of corona's, in the next place we are to diſcourſe upon parhelia and paraſelenæ or mock-ſuns and mock-moons; in which there is a greater variety of wonderful appearances, as may be perceived by the figures annexed. For beſides many ſurpriſing

[a] *Snellius* in his Book of the Comet in 1618.

priſing circles, there is not only a parhelion on each ſide of the ſun, (which two were the only ones obſerved in former ages) but alſo two or three, and ſometimes four more; as in the phænomenon obſerved by *Hevelius* in the year 1661. One may wonder how it ſhould be, that *Ariſtotle*, and *Cardan*, who wrote ſo many ages after him, ſhould affirm, that there never appeared above two parhelia together. Since it is not probable that ſix or ſeven parhelia ſhould often appear together within a few years, and that the like ſhould have never happened for ſo many ages before. The reaſon may be, that the two lateral parhelia which are always the brighteſt, were only taken notice of as parhelia, and that the reſt which are more languid and faint were overlooked. But by more accurate obſervers all thoſe lights are called parhelia which keep fixt in a certain place, though ſo faint in appearance as by the unſkilful to be taken for little white clouds. Now though there be a great variety of parhelia obſerved at different times, yet there are ſome things common to them all. I ſhall therefore attempt the ſolution of the moſt remarkable phænomenon that has been obſerved, and mention others by the by ſo far as any thing occurs that is common to all; leaving what is new and extraordinary in other phænomena to be diſcuſſed afterwards. I deſign therefore to conſider the Roman Phænomenon obſerved there by *Scheiner* the 20th of March 1629; which *Des Cartes* and *Gaſſendus* wrote upon about that time, from whoſe writings I take the following deſcription.

VIII. The *Roman* Phænomenon. Fig. 488.

534. A *the place of the obſerver at* Rome, B *the vertex or point over his head,* C *the true ſun,* AB *a vertical plane paſſing through the obſerver's eye, the true ſun and the vertex* B; *which are all projected in the ſtraight line* ACB. *About the ſun* C *there appeared two concentrick rings not compleat, but diverſified with colours. The leſſer and inner of them* DEF, *was fuller and more perfect; and though it was open from* D *to* F, *yet theſe ends* D *and* F *were perpetually endeavouring to unite; ſometimes they did unite and compleat the ring and then opened again. The other exterior and fainter and ſcarce diſcernable circle was* GKI; *it had a variety of colours but was very inconſtant. The third circle* KLMN *was very large and all over of a white colour, ſuch as are often ſeen with paraſelenæ about the moon. This was an excentrick circle paſſing through the middle of the ſun, at firſt entire, but towards the end of the appearance it was weak and ragged and ſcarce diſcernable from* M *towards* N. *In the common interſection of this circle and of the outward Iris* GKI, *there broke out two parhelia* N *and* K *not entirely perfect,* K *was ſomewhat weak, but* N *ſhone brighter and ſtronger. The brightneſs in the middle of them both reſembled that of the ſun, but towards their edges they were tinged with colours like thoſe of the rain-bow. They were not perfectly round and even at their edges, but uneven and ragged. The parhelion* N *was a little wavering, and ſent out a ſpiked tail* NP *of a colour ſomewhat fiery, which had a continual reciprocation.*

ſation. The parbelia at L *and* M, *beyond the zenith* B, *were not ſo bright as the former, but rounder and white like the circle which they were placed in; they reſembled milk or clean ſilver: the parbelion* M *was almoſt quite extinct at half an hour paſt two o' clock, excepting that ſome faint remains would revive now and then; and the circle it ſelf vaniſhed in that place. The parbelion* N *diſappeared before* K *did, and while* M *became fainter* K *grew brighter and vaniſhed laſt of all.*

IX.
A fuller deſcription of this Phænomenon.
Oper. Tom 3. p. 652.

535. To underſtand perfectly how this appearance was, imagine the great white circle *KLMN*, that paſſed through the true ſun, to be parallel to the horizon, having its pole *B* directly over the ſpectator's head. For as to the appearance of the figure [in *Gaſſendus** and *Des Cartes*] in which the vertex *B* is put nearer to *LM* than to *KN*; I take it to be a miſtake in the drawing. Since in all other obſervations, wherein the poſition of this white circle is deſcribed, it is always repreſented in a ſituation parallel to the horizon: although a ſmall inclination of it is not inconſiſtent with our theory. The ſpectator *A* muſt therefore be ſuppoſed to ſtand directly under the middle point *B*, in ſuch a poſition that when he looks at the ſuns *C*, *K*, *N*, he may have the other two at *L* and *M* behind his back.

X.

536. It is farther to be obſerved, that the order of the colours mentioned in the circles *DEF*, *GKN*, was the ſame as in the corona's abovementioned: namely the red next the ſun, as appears by *Scheiner*'s deſcription of the phænomenon in the year 1630 to be ſeen in the appendix. From which it appears that the diameter of the inner circle was alſo about 45 degrees, which is the uſual bigneſs of a corona: though I am apt to think that the exterior circle in the phænomenon of 1629 was leſs than the exterior circle in the phænomenon of 1630, as appears by the figure and may be proved. In this figure it is alſo to be obſerved that the tail of the mock-ſun *N* muſt be drawn within the great circle *KLMN*, as *Hevelius* rightly obſerved in the year 1661, which phænomenon may be ſeen below in the 561ſt article. But *Scheiner* has drawn it ſo, to ſhew it was extended from the ſun. The true ſun and the mock-ſuns are alſo drawn too big in compariſon to the circles, and the circles too broad for their diameters. For the diameter of the ſun ſhould be but about the 90th part of the diameter of the corona *DEF*, or of 45 degrees; and the breadth of the circle *KLM* ſhould be almoſt equal to that of the ſun. For by the explication of the phænomenon in 1630 it appears that the breadth of that circle was not equal to the ſun's diameter.

XI.
Cauſed by reflections and refractions through little cylinders.

537. Having conſidered the deſcriptions of theſe wonderful appearances let us now inquire into their cauſes. Though I ſoon perceived that globules of water, or partly of water and partly of ſnow, could not produce theſe effects, yet conſidering that parhelia are always attended with corona's, I was ſatisfied that their cauſes muſt be much alike. Conſidering there

therefore what other figures hail-ſtones might poſſibly have in the air beſides a ſpherical one, I found no other ſo ſimple as that of a cylinder. And indeed I had often obſerved that ſnow conſiſted of ſeveral ſlender oblong particles mixt with thoſe of other ſhapes; and ſeeing that ſmall globules were ſufficient for the production of corona's, I imagined that a great many little cylinders, floating in the air, might produce ſome effects of the like ſort. I remembered alſo that *Des Cartes* had taken notice of certain ſmall columns or cylinders which he had ſeen upon the ground, whoſe extremities were bounded with flat ſtars conſiſting of ſix rays or points. Having therefore conſidered the poſition of theſe cylinders, and that they could not eaſily be generated in any other but an upright poſture; and that they muſt ſometimes be partially thawed by the warmth of the ſun or air as well as the round grains; I ſoon found that all the particulars of the Roman Phænomenon might be produced by theſe ſmall cylinders; and was plainly convinced of it, after I had filled a hollow cylindrical glaſs with water, and had ſuſpended a ſolid cylinder in the middle of it. For by holding it in various poſitions to the ſun and to my eye, I found the large white circle was produced by reflection of the ſun's rays from the ſides of ſuch cylinders, and the parhelia on each ſide of the ſun by a double refraction through the ſides of them, and the other parhelia partly oppoſite to the ſun by a double refraction and an intermediate reflection, in the manner to be explained hereafter.

XII. How half-thawed cylinders may be generated.

538. But ſomething muſt be premiſed concerning the generation, poſition, figure and magnitude of theſe cylinders. As the globules which produce corona's muſt be very ſmall, perhaps ſmaller than turnepſeed, to be ſuſtained by aſcending exhalations, ſo I ſuppoſe theſe cylinders to be exceeding ſmall and ſlender for the ſame reaſon; which minuteneſs may be ſo far from hindering the perfection of their figures, that probably it may rather contribute to it. Theſe cylinders as well as the globules muſt be formed at firſt of the ſofteſt and fineſt particles of ſnow; that is out of the moſt minute and almoſt inviſible particles of a congealed cloud; (for ſnow is nothing but vapour congealed;) now ſo ſoon as a globule is formed, by a collection of theſe particles, it naturally follows that many more particles will ſoon adhere to the bottom of it, but not to its ſides. For ſince the particles of the cloud are driven upwards by a current of aſcending vapours, and ſince on the other hand the globules tend downwards by their weight or at leaſt aſcend ſlower than the vapours; it naturally follows that the particles of the cloud which ſtrike upon the bottom of a globule will ſtick to it and by degrees change it to an oblong cylindrical figure, while other particles of the cloud will eaſily ſlip by the ſides of it. And when a vaſt multitude of theſe cylinders are thus produced at ſmall diſtances from one another, it is probable enough, that the current of aſcending air and vapours, by paſſing between them, will preſerve them in

the ſame upright poſture in which they were formed: but when they are diſperſed by the wind or otherwiſe, they cannot continue upright, but will be turned into all manner of poſitions. Now when the warmth of the ſun or of the air ſhall have melted the outſides of theſe cylinders, in the ſame manner as was mentioned of the globules, a ſmaller cylinder of ſnow will remain in the middle of each of them ſurrounded with water; and after a certain part is melted, the cylinders within will become round and perfect, and will remain in this ſtate for ſome time, for the ſame reaſons that we gave for the globules. Nevertheleſs if that coat of water ſhould ſoon be frozen, it may poſſibly remain ſufficiently tranſparent and polite to tranſmit, refract and reflect the rays of the ſun in the regular manner to be deſcribed hereafter.

XIII. The large white circle how cauſed.

539. To begin the explanation of the Roman Phænomenon, firſt I ſay that the large white circle, which appeared in it, was produced by the reflection of the ſun's rays from the outſides of the upright cylinders. For when the ſun ſhines upon a vaſt number of ſuch cylinders ſuſpended in the air; it follows that a great white circle muſt appear to paſs through the ſun and to be parallel to the horizon, and to be equal in breadth to the breadth of the ſun. To make this appear, let us ſuppoſe a larger cylinder inſtead of one of thoſe ſmall ones, and conſider in what manner the rays of the ſun are reflected from it. Upon the upright cylinder *ABCD* let a ray *EF* coming from the center of the ſun, be reflected into the line *EG*: I ſay that *EF* and *EG* are equally inclined to the horizon. For let *HEK* be a line drawn through *E* upon the ſide of the cylinder, and imagine a plane *LI* to touch the cylinder along the line *HEK*; and the ray *EF* will be reflected from this plane in the ſame manner as from the cylinder[a]. Now ſuppoſing another plane paſſing through the rays *FE*, *EG* to ſtand perpendicular to the plane *LI*, upon the line *MEN*, the angles of incidence and reflection *FEM*, *GEN* will be equal[b]. Imagine a ſpherical ſurface whoſe center is *E* to cut off equal lines *EF*, *EM*, *EO* and alſo *EG*, *EN*, *EP*, and let *EO* and *EP* be parts of the line *HEK*; and let *FO*, *FM*, *OM* be arches of great circles in that ſpherical ſurface and likewiſe *GP*, *GN*, *PN*. Therefore ſince the plane *FMNG* ſtands at right angles to the plane *LI*, and ſince both of them paſs through *E* the center of the ſphere; in the ſpherical triangle *FMO* the angle *M* will be a right one, and likewiſe the angle *N* in the triangle *GNP*. But the ſide *MO* is equal to *NP*, becauſe the angle *MEO* is equal to *NEP*; and likewiſe the ſide *MF* is equal to the ſide *GN* becauſe the angles *FEM*, *GEN* are equal. Therefore the remaining ſides *FO*, *GP* are equal, and by conſequence the angle *FEO* is equal to *GEP*; whoſe complements to a right angle are alſo equal, and meaſure the inclinations of the rays *FE*, *EG* to the horizon; which was to be proved.

Fig. 489.

[a] Art. 19.

[b] Art. 8.

The demonſtration is the ſame when the ray *FE* is reflected from the inſide of the cylinder; by which I ſhall ſhew that the parhelia muſt appear in the great white circle.

540. Hence it appears, when the air is full of thoſe minute upright cylinders, that the rays proceeding from the ſun's center or from any other point of his diſk, will be reflected from thoſe cylinders towards the horizon in the very ſame angle in which that point of the ſun is elevated above the horizon. Becauſe by reaſon of the ſun's immenſe diſtance, every angle of incidence *FEO* upon every cylinder, whether higher or lower, is of the ſame magnitude. Conſequently the ſpectator below can ſee the ſun reflected from thoſe cylinders only, from which a line drawn to his eye makes an angle with the horizon equal to the altitude of the ſun. Therefore if lines be drawn every way from the eye of the ſpectator making equal angles with the horizon to the ſun's altitude, it is certain that all of them together will trace out a circle in the heavens parallel to the horizon. It is plain therefore that the ſpectator will perceive a white circle parallel to the horizon and paſſing through the ſun. It is alſo eaſy to be underſtood that the breadth of this circle will be equal to the breadth of the ſun. For every point of the ſun's vertical diameter, as well as his center, will illuminate a circle of cylinders of the ſame apparent height as the illuminating point. By which means the upper and under edges of the broad white circle will be parallel, and at the ſame diſtance from each other as the uppermoſt and undermoſt points of the ſun. It is alſo remarkable that while the ſun aſcends or deſcends, this circle will alſo aſcend or deſcend, and ſo will become bigger or leſs. And moreover that diverſe ſpectators, though never ſo remote from one another, will every one ſee a different circle paſſing through the ſun, in like manner as in the rainbow. But this cannot be true of that circle which *Des Cartes* propoſes to us as the cauſe of this phænomenon. For he ſuppoſes it to be a large ſolid ring of ice ſuſpended in the air, which cannot poſſibly be directly over the heads of ſeveral diſtant ſpectators. Beſides that no reaſon can be given for its paſſing through the ſun, ſometimes for two or three hours together, as we find in fact that it does. It is alſo obſervable that no thick clouds are ſeen in the air when theſe circles appear, but ſuch only as are very thin and ſcarce viſible. For we find it taken notice of in moſt of theſe obſervations, that the ſky was very clear and ſerene. Which is no wonder in our hypotheſis, conſidering that theſe minute cylinders conſtitute a very thin cloud, uniformly extended, through which the ſun and even the blue colour of the ſky may be ſeen. It happens ſometimes that ſome parts of this circle appear faint and ſcarce viſible, for no other reaſon but want of matter or a ſufficient quantity of cylinders: excepting that the part within the halo, though there be matter enough, is leſs diſcernable, and not taken notice of in ſome obſervations, by reaſon of the XIV.

neſs of the neighbouring ſun; and chiefly becauſe the white circle is much brighter juſt without the circumference of the corona, than in other parts of it. The reaſon of which will be underſtood when we explain the lateral parhelia at *K* and *N*.

XV. The lateral parhelia explained.

541. As to theſe parhelia at *K* and *N*, I ſay they are produced by the ſame upright cylinders, which produce the great white circle, by means of a double refraction of the rays of the ſun, exactly in the ſame manner in which corona's are produced by globules with ſnow in their centers. For becauſe a certain part of each of theſe cylinders is thawed, and contains a leſſer cylinder of ſnow within it, the ſun cannot be ſeen through thoſe cylinders which conſtitute the part *KN* of the white circle, but only through thoſe which lye without it, for the ſame reaſon as in corona's. From hence it alſo comes to paſs that the interval between theſe parhelia becomes ſo much greater as the internal cylinder of ſnow is thicker in proportion to the whole cylinder. But the ſun appears brighteſt of all through thoſe cylinders on the outſides of the arch *KN*, that adjoin to it, and his brightneſs decays gradually in going outwards from thence to a certain diſtance; and this is the cauſe of the tails of theſe parhelia. But ſince the colours appear moſt vivid in thoſe cylinders that are next to the outſide of the arch *KN*, for the ſame reaſon as in corona's; theſe colours make the parhelion diſtinct from the reſt of the tail, which is leſs bright and very little coloured; in like manner as the corona and the rain-bow have but a ſmall breadth, although they are terminated but on one ſide. Now theſe tails and alſo the parhelia at the head of them, always tend towards the white circle (as will appear preſently) and make it brighter ſo far as they extend. The tail *NP* of the parhelion at *N* is drawn as appearing out of this circle, but this was either a miſtake or was deſign'd to ſhew that the tail tended directly from the ſun: as *Hevelius* obſerves more accurately in the phænomenon in 1661*. Therefore though it is not related that the parhelion *K* had a tail, nevertheleſs it is plain that a certain part of the white circle was its tail, though but a dull one; becauſe this parhelion is ſaid to be fainter than the other. And it appears by *Hevelius*'s obſervations as well as my own, that all parhelia and alſo paraſelenæ or mock-moons, on each ſide of the moon, are conſtantly adorned with tails. Laſtly the extraordinary brightneſs of the parhelia, which was ſaid to rival that of the ſun[a], will be eaſily accounted for, by conſidering that every cylinder ſhines throughout its whole length; whereas the round grains in the corona or in the rain-bow emit but a little light, ſo that one cylinder may probably afford more light than ten round grains. Therefore if there be great plenty of cylinders in the air, it is no wonder that theſe images of the ſun appear ſo bright. I ſhall now demonſtrate theſe aſſertions, by an accurate conſideration of the refraction of the rays through the ſides of the cylinders. And firſt I ſhall ſhew that theſe parhelia and their tails muſt neceſſarily appear in the great white circle.

* Art. 561.

[a] Art. 534.

542.

XVI. Why the lateral parhelia are in the white circle. Fig. 490.

542. Let *ABCD* be one of theſe cylinders hanging upright in the air; and let the ray *EF* fall upon its watery ſurface, and be refracted into the cylinder along the line *FG*, and at *G* let it emerge by refraction through the watery ſurface in the line *GH*; I ſay this ray *GH* will be inclined to the horizon in the ſame angle as *EF* is inclined to it; that is in an angle equal to the ſun's altitude. For let *ABCD* be a plane parallel to the axis of the cylinder and paſſing through the points *F*, *G*; and it will cut the ſurface of the cylinder in equal angles in the lines *AB*, *CD* parallel to each other. And conſequently ſince the ray *FG* is ſituated in this plane between the parallel lines *AB*, *CD*, it will be equally inclined to the cylindrick ſurface towards *F* and *G*, ſo as to make the angles *GFC*, *FGA* equal to one another. Hence it is too plain to need any demonſtration, that the refracted ray *GH* tends downwards in the ſame angle, as in going backwards along *GF* it would tend upwards along *FE* *; that is the angles which *GH* and *EF* do make with the ſides of the cylinder are equal. Imagine two planes to be drawn, one through *CD* and the ray *EF*, and the other through *AB* and the ray *GH*; and it evidently follows that theſe two planes are equally inclined to the intermediate plane *ABCD* in which *FG* is ſituated and that the angles, which meaſure their inclinations to this middle plane, are *KCB*, *LBC*, being made by the interſections of theſe planes with the baſe of the cylinder; and conſequently that theſe angles *KCB*, *LBC* are equal. And ſince we have ſhewn that the angles *DFE*, *BGH* are equal, it appears from hence alſo that the incident ray *EF* and the emergent ray *GH* after both refractions, are equally inclined to the horizon. Therefore the ſuns rays ſo tranſmitted through theſe cylinders, cannot come to the ſpectator's eye but from thoſe cylinders, whoſe apparent altitude is the ſame as that of the ſun; or only from thoſe very cylinders in which the great white circle appeared by reflected rays: and conſequently the two parhelia produced by ſuch refractions muſt appear alſo in that white circle.

XVII. The parhelia appear at various diſtances from the ſun, and why. Fig. 490.

543. To determine at what diſtance from the ſun the parhelia ought to appear, we muſt attend to that ray of the ſun which, in paſſing through the water, touches the internal cylinder of ſnow. For ſuppoſing *FG* to be that ray, then *BC* will alſo touch the ſaid cylinder in the plane of its baſe. Therefore drawing *OMN* in the plane of the baſe, through its center *N* and parallel to *KC* which lyes under the incident ray *EF*, let it meet the line *BL* in *M* which lyes under the emergent ray *GH*; and the angle *BMN* will be equal to the angle which meaſures the inclination of two vertical planes, one of which paſſes through the ſun and the other through the parhelion; and both of them through the eye of the ſpectator. And conſequently in fig. 488 the part *CN* of the white circle, intercepted tween the ſun and one of the lateral parhelia, will contain the ſame number of degrees of its circle, as are contained in the angle *BMN*. Now this angu-

* Art. 15.

angular diſtance of the lateral parhelion from the ſun, is ſo much the greater as the internal cylinder of ſnow is thicker in proportion to the whole cylinder: and moreover ſuppoſing the ratio of the diameters of theſe cylinders to be invariable, that angular diſtance becomes greater, as the ſun riſes higher: as will appear by the following table, whoſe conſtruction, which would have been too tedious here, may be ſeen in the appendix to this chapter.

I.	II.		III.		IV.	
Degr.	deg.	min.	deg.	min.	deg.	min.
0.	22.	00	22.	30	45.	00
5.	22.	10	22.	38	45.	26
10.	22.	38	23.	08	46.	44
15.	23.	28	24.	00	49.	04
20.	24.	42	25.	16	52.	46
25.	26.	26	27.	04	58.	24
30.	28.	48	29.	26	67.	42
35.	31.	58	32.	42	94.	22
40.	36.	18	37.	10	The parhelion can be ſeen no farther in this cylinder.	
45.	42.	18	43.	14		
50.	51.	00	52.	26		
55.	64.	48	66.	64		
60.	92.	34	98.	24		

The 1st column ſhews the ſun's altitude; the 2d the angle made by two vertical planes, one paſſing through the ſun and the other through the lateral parhelion, when the diameter of the whole cylinder is to the diameter of the internal one as 1000 to 473; the 3d column ſhews the ſame angles between the verticals when the ratio of the diameters is 1000 to 480; the 4th when that ratio is 1000 to 680. Suppoſing this ratio as 1000 to 714, then the ſaid angle of the verticals will be 88 deg. 48 min. when the ſun is 25 degrees high; and can be no bigger in this altitude; and this relates to the two parhelia obſerved by *Hevelius*[a].

[a] Art. 561.

XVIII. That ſeveral parhelia with tails may appear together.

544. If a ſufficient number of cylinders be placed above one another ſome of which are melted more and ſome leſs, it appears by theſe tables, that beſides the collateral parhelia next the ſun, two or more may appear farther from him, but ſtill in the ſame white circle: which is alſo confirmed by the obſervations of *Hevelius* in the year 1661. Feb. 20. and of *Scheiner* in the year 1630; of which more hereafter. It appears alſo from the ſame tables, that while the cylinders remain in the ſame ſituation, and the ſun riſes higher, the intervals between him and the two parhelia will increaſe, and will decreaſe again while he deſcends lower: the very thing that I obſerved and ſhall mention in my account of that obſervation. But a greater alteration of theſe intervals may happen if the cylinders be farther melted: and from hence there aroſe a very remarkable phænomenon in former days. For *Julius Obſequens* relates *that in the time of Auguſtus Cæſar, when M. Lepidus and Munatius Plancus were Conſuls, there appeared three ſun's together, which ſoon after were contracted into one orb.* This is ſuch an appearance as I never met with any where elſe; but the cauſe of it is eaſily underſtood, and is nothing elſe but the warmth of the air; which increaſing upon the former temperature of it might preſently

melt all the ſnow in the cylinders, and by this means both the parhelia would draw nearer to the ſun till they coincided with it, when the whole cylinders were melted and changed into round drops. And ſeeing this agrees ſo exactly with our theory, it cannot be doubted but we have taken the words of *Obſequens* in their true ſenſe; although *Gaſſendus* has endeavoured at another interpretation[a]; as if the parhelia appearing at firſt, there preſently broke out a corona which contained them within it. For then it would not be ſo proper to ſay that three ſuns were contracted into one orb, but two rather; neither would this have been ſo uncommon a ſight. Beſides this, they imagined that a repreſentation of the Triumvirate was ſignified by this prodigy; as appears by the following paſſage of *Dion Caſſius* compared with that of *Obſequens*. *Dion* ſays, book 45, that the light of the ſun appeared ſometimes to be diminiſhed and extinguiſhed, and ſometimes to break out again in a triple orb; (for ſo I interpret his words τότε δὲ ἐν τρισὶ κύκλοις φαντάζεσθαι) one of which was ſurrounded by a radiating fiery crown; by which future events were foretold. We might accurately determine the lengths of the tails of theſe parhelia; but the farther they recede from the parhelia the fainter they grow, and therefore the true extent of them cannot well be obſerved. It is therefore ſufficient to know that they are extended throughout a whole quadrant of the white circle beginning from the true ſun; and ſomewhat farther, as the ſun's altitude may happen to be greater. But yet their light is frequently ſo weak as by obſervers to be taken only for a part of the ſaid circle.

[a] Op. Tom. 3. p. 655.

XIX. The corona paſſing thro' the parhelia is not made by globules.

545. I am now going to conſider thoſe corona's which almoſt always appear with theſe lateral parhelia. For though corona's are ſometimes ſeen without parhelia, yet parhelia never appear without a corona paſſing through them, unleſs it be ſo weak ſometimes as to eſcape obſervation; and this appearance being not accidental muſt proceed from a certain cauſe. When ſmall cylinders are produced in the air, it is probable that half-thawed globules are lodged there at the ſame time, for the production of corona's. But yet it is difficult to conceive by what cauſe they can be thawed in ſo exact a proportion, as to produce a corona that ſhall paſs preciſely through theſe very parhelia. For ſuppoſing the round and oblong hail to be thawed exactly alike in diameter, neverthleſs it appears by the foregoing tables that the parhelia would lye on the outſide of the corona. For when the ratio of the diameter of the outward to that of the inward cylinder is as 1000 to 473, the diſtance of the parhelia from the ſun at various altitudes will be 22, 28, 36, 51, and any more degrees of the white circle, though the ſemidiameter of the corona be but 22 at all altitudes of the ſun, when the globules are thawed in the ſame ratio. This alſo is farther to be obſerved that while the parhelia continually approach towards each other when the ſun deſcends, and recede from each other when he aſcends, as we obſerved above; the corona ſhould alſo contract

and enlarge it ſelf, ſince it always paſſes through the parhelia; but in the round grains no cauſe can be aſſigned for ſuch an effect.

XX. Nor by cylinders obliquely ſituated.

546. But beſides the round grains, it is certain that thoſe very cylinders, which I obſerved may ſwim in the air in uncertain poſitions, are capable of producing corona's. For by reaſon of the ſnow within them they do not tranſmit the rays of the ſun, unleſs they lye wide of a certain diſtance from him; which angular diſtance is determined by thoſe cylinders on whoſe ſides the rays of the ſun fall perpendicular; for the reſt of them upon which the rays fall obliquely muſt be ſtill remoter from the ſun to tranſmit his rays to the eye: as appears by the tables above, taking the altitude of the ſun in the table for his altitude above the plane of the baſe of theſe cylinders. So that theſe cylinders as well as the globules may produce a corona bounded with a red colour on its ſide next the ſun. And probably the corona's that appear with the parhelia may proceed from this cauſe. Firſt becauſe theſe little cylinders are alone ſufficient. Secondly becauſe the corona's that appear with parhelia are often brighter coloured than without parhelia. Becauſe the cylinders ſend rays from their whole lengths in greater plenty than the round grains, which ſend them as it were but from a point. But to return to the purpoſe, there is no reaſon at all why this corona produced by cylinders ſhould paſs through the parhelia rather than that which proceeds from round grains. And the ſame difficulty in the alteration of the diſtance of the parhelia ſtill remains as before. There is no reaſon therefore to maintain that the corona which paſſes through the parhelia is produced by thoſe cylinders that are placed in an uncertain order. Indeed as to the Roman Phænomenon fig. 488 I do not doubt but the corona *DEF*, within the parhelia, was produced by cylinders diſpoſed in the manner abovementioned. But ſtill as to the other part of the phænomenon there ſeems to be ſome difficulty.

XXI. But by the round ends of upright cylinders.

547. Nevertheleſs we may be able to conquer this difficulty by attending more cloſely to the figure of thoſe cylinders. Hitherto we have only conſidered their ſides; now let us ſee what ſort of extremities they are likely to have. To ſuppoſe them terminated by plane baſes, is by no means agreeable to nature. But becauſe their tops and bottoms are thawed as well as their ſides, and to the ſame degree quite round, moſt certainly the water at their extremities will affect a round figure; whence alſo the inner cylinder of ſnow will be formed at each end either into an hemiſphere or an half ſpheroid; and then the ſhape of the whole cylinder with the included kernel ought to be ſuch as this figure repreſents in a larger ſize. Hence it will come to paſs, that not only thoſe upright cylinders, which lye on each ſide of the ſun and produce the collateral parhelia, will tranſmit his rays to the eye, but alſo thoſe that lye above and below and quite round him beyond a certain diſtance; which therefore will produce a corona. Through thoſe above him the rays will paſs as the ray *DCBA* does and

Fig. 491.

and will touch the top of the kernel in H; and the rays will paſs in like manner through the bottom of the lower cylinders as $EFGA$ does, and will touch the bottom of the kernel in K. But ſince the exact figure of the convexities BCL and FGM is unknown, it would be in vain to attempt an exact determination of the quantity of refraction of the rays that paſs through it; and much more of thoſe that paſs towards the ſides of theſe oval ſolids. Nevertheleſs having ordered a glaſs cylinder to be made in the ſhape repreſented by half this figure; and having filled it with water and ſuſpended an opake cylinder within it, that might in all places be equally diſtant from the glaſs; and having placed this compound cylinder in various poſitions between the ſun and my eye, I found by experience it tranſmitted the rays in ſuch manner that the angles BAN, GAF were nearly equal to one another and to all other angles under the rays that touched the opake cylinder in paſſing by it; taking the line AN for a line drawn from the eye to the ſun; or for a line parallel to it.

548. Now ſince the little cylinders in the clouds muſt have the ſame power and property as this fictitious one, they will form a corona which will be nearly circular. So that theſe upright cylinders, which formed the great white circle by reflection, will alſo form thoſe corona's by refraction that paſs through the parhelia. Conſidering therefore the neceſſity we have ſhewn for the ſhape of theſe cylinders, and that the rays paſſing through their extremities will produce coloured circles, I do not doubt but theſe circles are thoſe very corona's; otherwiſe ſome irregular and ill-ſhaped circles muſt neceſſarily be viſible. And though the eye may not be able to diſcern a ſmall defect of roundneſs, yet it may happen that one diameter of a corona may be different from another. Which I find is confirmed by obſervation. For in an obſervation made at *Rome* in the year 1630, when two circles about the ſun are ſaid to croſs each other above and below him; and that parhelia ſhone out from the outward arches on each ſide of him; it ſeems more probable to me that inſtead of two circles croſſing each other, we muſt firſt conceive a circle pretty oblong, in which two parhelia were produced by the upright cylinders, and then another circle not quite ſo oblong, touching the other above and below, produced either by cylinders in a confuſed ſituation or by round grains. For the figure made by theſe oblong circles is ſo very like the figure made by two perfect circles, as not eaſily to be diſtinguiſhed from it. And beſides it is exceedingly probable that one and the ſame circle contained two parhelia. XXII. Fig. 492.

549. The circle that paſſes through the parhelia frequently appears fainter above and below, and brighteſt next the parhelia; as *Hevelius* has accurately remarked in many obſervations. The reaſon of this is alſo plain by our glaſs cylinder. For the rays paſſing through the round tops and bottoms of them produce only a ſmall round image of the ſun, as round grains

XXIII. The parts of this corona are brighteſt next the parhelia.

grains do; but thoſe which paſs towards the ends of the cylindrical ſurfaces where they juſt begin to turn roundiſh are formed into a broader and brighter pencil; in the ſame manner as thoſe are which produce the parhelia; which makes the corona grow gradually brighter and brighter towards the parhelia both above and below, and more intenſely coloured.

XXIV. The back parhelia *L*, *M* how produced.

550. Having explained the cauſes of the two corona's in the Roman Phænomenon, let us proceed to account for thoſe two parhelia *L*, *M* which appeared in the back part of the great white circle. Theſe I ſay, are produced alſo by the upright cylinders; by the ſame refractions of the ſun's rays which produce the rain-bow. It muſt be obſerved and ſhall be demonſtrated by and by, that the inner opake cylinders are ſo far from contributing to the production of theſe parhelia, that ſometimes they hinder their appearance. But yet the inner cylinder of ſnow is abſolutely neceſſary, whether the outward ſhell that contains it be made of water or ice. For without the ſnow the watery cylinders would immediately be changed to round drops; and the whole cylinders cannot be all ice, becauſe we cannot conceive them to be every where tranſparent, without ſuppoſing them quite liquid before they were frozen; which I have juſt now ſhewn to be inconſiſtent with a cylindrical figure.

XXV. The baſes of the rain-bow anſwer to two back parhelia when the ſun is in the horizon. Fig. 493. a See Art. 494. b See the remarks upon chap. 10.

551. But before we conſider the manner of refraction through cylinders, it will be convenient firſt to conſider it in round drops, to ſhew the cauſe of the rain-bow, which *Des Cartes* firſt [a] diſcovered. Let *ABC* be a globule of water, *DA* a ray of the ſun falling upon it, and refracted from *A* to *B*, and then reflected from *B* to *C*; ſo that *AB* and *BC* may be equal; and at *C* let it be refracted again towards the eye at *E*. Again drawing *CH* and *EF* parallel to the ray *DA*, and making the angle *FEG* equal to *FEC*, it appears by *Des Cartes*'s tables [b], that the angle *HCE* will vary its magnitude according as the ray *DA* is ſuppoſed to fall upon different parts of the drop; but ſo as never to exceed 41° 30′; and therefore the angles *CEF* and *FEG* never exceed that quantity. Hence it follows that no drops that lye out of the cone *CEG* can tranſmit any rays to the eye; and of thoſe that lye within it, the neareſt to the conick ſurface will tranſmit them moſt plentifully; and by this means the brighter drops which boarder upon the dark ones will be more apparent and will make a round ring; which would be viſible quite round but for the ſurface of the earth; whereas one half at moſt can only be ſeen when the ſun is in the horizon. This is the true cauſe of the rain-bow, which any one may examine experimentally, by expoſing a glaſs globe full of water in the ſame poſition to the ſun as the globule *ABC*. Suppoſe now the circle *ABC* to be an horizontal ſection of an upright cylinder, and that innumerable others are placed upright upon the plane of the horizon, and that the rays of the ſun being alſo in the horizon are refracted through them in the ſame manner as through the drop *ABC*, ſo as to make the greateſt

angle *HCE* or *CEF* equal to 41° 30′: and it follows that no cylinder out of the angle *CEG* can tranſmit a ray to the eye; and that thoſe which are neareſt to the ſurface aforeſaid will tranſmit the moſt rays. Therefore a ſmall portion, like the baſe, of a rain-bow will appear in the cylinders towards *C*, and another likewiſe towards *G*; and the intercepted arch of the horizon will contain as many degrees as the angle *CEG* does, that is 83°. Thus we have two parhelia much brighter than the parts of the rain-bow, as being produced by cylinders, which tranſmit more rays than globules do, as we have often obſerved before. Thus much for the appearance of parhelia when the ſun is in the horizon.

XXVI. Why the back parhelia appear in the large white circle. Fig. 494.

552. But the ſun being elevated, theſe two back-parhelia will appear in the white circle. To demonſtrate this, let *ABCD* be an upright cylinder in the air; *EF* a ray of the ſun falling upon it; *FG* the ſame ray refracted into the cylinder, *GH* the ſame ray reflected from its ſurface at *G*, and *HK* the ſame ray emerging by refraction at *H*: I ſay the incident and emergent rays *EF*, *GH* will be equally inclined to the plane of the horizon.

XXVII. Demonſtration.

553. For by the laws of reflection the ray *FG* will be ſo reflected into the line *GH*, that two planes drawn through *FG*, *GH* parallel to the axis of the cylinder, will cut the baſes and ſides of the cylinder in two parallelograms *PAQC*, *ABDQ* equal to each other in length and breadth; and the rays *FG*, *GH* will be equally inclined to the ſides of theſe parallelograms. This is plain enough by what has been demonſtrated and obſerved at the end of the 539th art. Therefore ſince the rays *FG*, *GH* are equally inclined to the lines *PC*, *BD*, the ray *GH* in going forward will emerge by refraction at *H* along *HK*, and *GF* ſuppoſed to go backwards will emerge at *F* along *FE** in ſuch manner that the angles *PFE*, *DHK* will be equal. Now let the line *CL* lye in the plane of the baſe directly under *EF*, and alſo *DM* under *HK*; and the angles *QCL*, *QDM* will be equal; and ſince the angles *PFE*, *DHK* are equal, the rays *EF* and *HK* will be equally inclined to *CL* and *DM*, that is to the plane of the horizon. Therefore the ſun's rays tranſmitted along theſe lines that bend about the cylinders hanging in the air, cannot come to the ſpectator's eye, but from thoſe cylinders only whoſe apparent height above the eye is equal to the apparent height of the ſun; that is from thoſe only which cauſed the white circle to appear by reflections from them. And ſo it is manifeſt that the parhelia thus produced can appear no where elſe but in the ſaid white circle.

* Art. 9.

XXVIII. The diſtance between the back parhelia determined. Fig. 494.

554. Now the reaſon why theſe parhelia appear in certain determinate places of the white circle, is altogether ſimilar to that which we gave for the rain-bow. For if in the plane of the baſe the lines *DN*, *MO* be drawn parallel to *CL* which lyes under the incident ray *EF*; I find that the angle *MDN* or *DMO* cannot exceed a certain determinate magnitude; and that its greateſt magnitude will vary while the ſun's altitude varies.

For example, when the ſun is 25 degrees high this angle is 33° 18′ at moſt; and ſince the rays which fall upon the cylinder emerge in the limit of this greateſt angle more copiouſly than when it is ſmaller; this greater quantity of rays is the cauſe that the parhelia appear on each ſide of *MO* at that angular diſtance. Drawing therefore *MR* in the plane of the baſe ſo as to make the angle *OMR* equal to *OMD*, the angular diſtance of the two parhelia will be equal to the whole angle *DMR*: ſo that two vertical planes ſtanding upon the lines *MD*, *MR* will paſs through the parhelia; and the arch of the horizontal circle and likewiſe that of the white circle parallel to it and intercepted between theſe planes, will contain exactly as many degrees as the angle *DMR* does. The number of degrees in half this angular diſtance is computed in the following table for ſeveral altitudes of the ſun. But the manner of making the computation, being too prolix for this place, is taught in the appendix to this chapter.

Degr.	deg.	min.
0.	41.	30
5.	41.	08
10.	40.	14
15.	38.	36
20.	36.	16
25.	33.	18
30.	29.	36
35.	25.	16
40.	20.	12
45.	14.	40
50.	8.	44
55.	3.	06
58.	0.	32

The ſecond column of this table ſhews the angle between a vertical plane, paſſing through the ſun, and another paſſing through one of the parhelia, anſwering to the ſun's altitude put over againſt it in the firſt column.

XXIX. The deſcription of the Roman Phænomenon corrected.

555. By this table I find the diſtance between the back parhelia of the Roman Phænomenon to be about 60 degrees. For the height of the pole at *Rome* being 42° 2′, the ſun's altitude there at three o' clock in the afternoon was about 30 degrees; and in this table half the diſtance of the parhelia anſwering to this altitude is 29° 36′. Indeed in *Scheiner*'s ſcheme as it is drawn in *Gaſſendus*'s little book[a], the diſtance of theſe parhelia exceeds 90 degrees; but it was neither meaſured nor eſtimated, nor ſo much as mentioned; and therefore I dare affirm it is repreſented too big in the figure. For the diſtance between any two points in the heavens appears ſo much the greater as the points are nearer to the horizon: juſt as the diſtances between the ſtars in the great Bear, for example, ſeem to be twice as big when they are near the horizon as when they are near the vertex. For the like reaſon the diſtance between theſe two parhelia ſeemed to bear a greater proportion than it really had, to an arch, which paſſing through the vertex, defined the magnitude of the diameter of the white circle *KLMN*. But this could not happen in the diſtance between the parhelia *K*, *N*, becauſe of the known magnitude of the circle *DEF*, whoſe diameter is 45 degrees. The ſame fallacy in the diſtance of the back parhelia will appear plainer in ſome of the following obſervations. It is alſo upon the ſame account, that the diſk of the ſun appears almoſt twice as big in the horizon

Fig. 488.

as

[a] See Oper. Tom. 3. p. 652.

as when he is higher elevated; and likewiſe that the rain-bow appears to be a part of the greateſt circle of a ſphere, whereas it is not half ſo big.

XXX. Why objects in the sky appear larger near the horizon than higher up.

556. Now the cauſe of this fallacy in ſhort is this; that we think the ſun or any thing elſe in the heavens to be remoter from us when it is near the horizon, than when it approaches towards the vertex. Becauſe we imagine every thing in the air, that appears near the vertex, to be no farther from us than the clouds that fly over our heads. Whereas on the other hand, we are uſed to obſerve a large extent of land, lying between us and the objects near the horizon, at the far end of which the convexity of the ſky begins to appear; which therefore together with the objects that appear in it, is uſually imagined to be much farther from us. Now when two objects of equal magnitudes appear under the ſame viſual angle, we always judge that object to be larger which we think is remoter. And this is the true cauſe of the deception we have been ſpeaking of.

XXXI. The colours of the back parhelia conſidered.

557. But to return to the parhelia; it muſt be obſerved, that in the Roman phænomenon they are ſaid to have appeared white, whereas they ought to have been coloured, for the ſame reaſon as in the rain-bow and in the glaſs cylinder filled with water. I take the cauſe of this whiteneſs to ariſe from the weakneſs of their light, as we ſaid before of corona's that ſometimes appear white[a]. This alſo happens to the lateral parhelia, as appears by the accurate obſervation of *Sam. Kechelius* to be ſeen in the appendix: in which the parhelion *C*, which was fainter than the other, is ſaid to change from yellow to white, which is the ſame as the ſilver colour obſerved in our parhelia. Now the yellow and white colours, which the cylinders tranſmit by the refractions aforeſaid, being far more ſplendid than the red, it muſt neceſſarily follow, that when but few cylinders remain, the red will diſappear before the yellow and white; and ſo the light of the parhelia will remain without any tincture of colours. But that theſe back parhelia do ſometimes appear coloured, is evident from the Engliſh obſervation, taken from *Math. Paris*'s Hiſtory, to be ſeen in the appendix. Where we are told that beſides the true ſun, four mock-ſuns appeared red in a great circle of the colour of chryſtal; and that two of theſe were back parhelia, is manifeſt from the figure, though faulty perhaps in other reſpects.

[a] Art. 530.

Fig. 503.

Fig. 502.

XXXII. Why theſe parhelia don't appear ſometimes.

558. This ſcarcity of cylinders may perhaps be the reaſon that in many obſervations theſe parhelia have not appeared, although the white circle was plain enough to be ſeen: as in a phænomenon at *Rome* in the year 1630 and in that ſeen by *Hevelius* in the year 1661 Feb. 20[b]; and in ſome others: in which the parhelia did not appear by reaſon of too faint a light. Neverthele ſs there is another thing which may hinder their appearance; namely if in the back part of the white circle, the inner cylinders contained in the water be too thick in proportion to the water; for then they will ſtop the rays which ſhould produce theſe parhelia. For I find when the

[b] Art. 561.

ſun is 25 degrees high, if the diameter of the opake cylinder ſhall bear a greater proportion to that of the whole than 590 to 1000, that no back parhelia can appear. Hence in a phænomenon obſerved at *Rome* in the year 1630, when the ſun was 28 degrees high, if the cylinders that produced the parhelia *M* and *N* did not extend ſo far as to the back part of the white circle, and no others lay there but ſuch as produced the parhelia *O* and *P*, no back parhelia could appear in that region: ſince in theſe cylinders the ratio of the internal to the external diameter was as 624 to 1000. Likewiſe in *Hevelius*'s obſervation when 6 parhelia appeared, the ſun being 25 degrees high, if the cylinders that produced the lateral parhelia *B* and *C* did not extend to the back part of the white circle, and no others lay there but ſuch as produced the parhelia *E* and *D*, the back parhelia could not appear. For the ratio of the diameters of theſe internal and external cylinders was 714 to 1000.

Fig. 498.

Fig. 495.

XXXIII. Why the back parhelia have no tails. Fig. 488.

559. The reaſon why the back parhelia in the Roman Phænomenon appeared rounder than the lateral ones is this. Although ſome cylinders in the arch *LM*, that lye not far from *L* and *M*, ſend ſome refracted rays to the eye, yet they are not near ſo numerous as thoſe which come from the cylinders at *L* and *M**; and therefore theſe parhelia have no tails that are far extended, like the lateral ones. Beſides this, the refractions through the cylinders that produce the back parhelia, are more regular. For the rays are not confined in their courſe by the opake cylinders within, but receive their directions from the refractions at the moſt exact figure and poliſh of the outward watery ſurface of the cylinders. For the dark cylinders within not being exactly equal to one another, are the cauſe of the irregular and inconſtant ſhape of the lateral parhelia. The reaſon of the reciprocation or fluctuating motion of the tail of the parhelion *N*, was that ſometimes fewer and ſometimes more cylinders were carried into that place; which is alſo the reaſon that the corona *DEF* was ſometimes entire and ſometimes open below, and that the parhelion *K* grew brighter while *N* grew duller.

* Art. 551. 552.

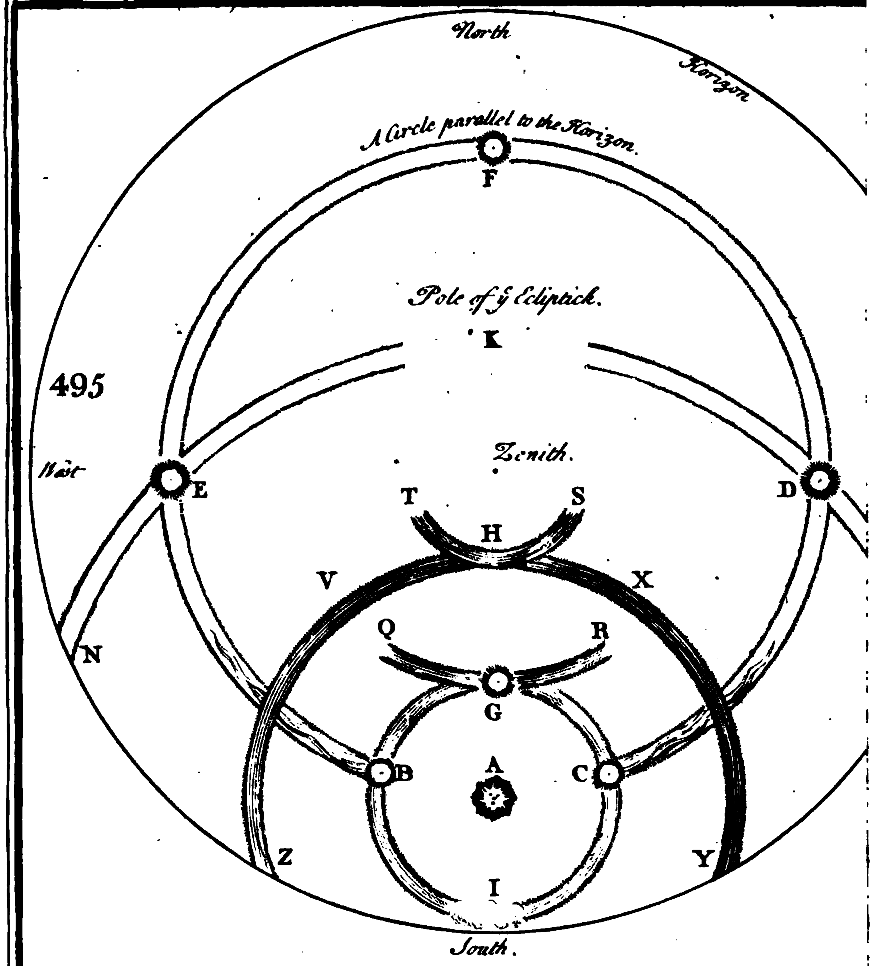
XXXIV.

560. I have now gone through the explication of every particular of the Roman Phænomenon, referring the cauſes of all the parhelia and circles in it to half-melted cylinders, partly upright and partly in a confuſed order; which cauſes, ſo agreeable to the phænomena and alſo to one another, do every where confirm one another; ſo that I think the reality of them cannot be doubted.

XXXV. Tranſition to an extraordinary phænomenon of 7 ſuns ſeen at *Dantzick* by *Hevelius*.

561. Having accounted for every particular in the Roman Phænomenon, let us now proceed to *Hevelius*'s obſervation, 20th Feb. 1661; in which more ſuns and circles were obſerved, and ſome of them in a different poſition from thoſe we have hitherto conſidered. But ſince the cauſe of them is not to be found either in the upright cylinders, or in thoſe that fly about confuſedly; (for our glaſs cylinder viewed in all poſitions diſcovers

495


North
Horizon
A Circle parallel to the Horizon
F
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K
West
Zenith
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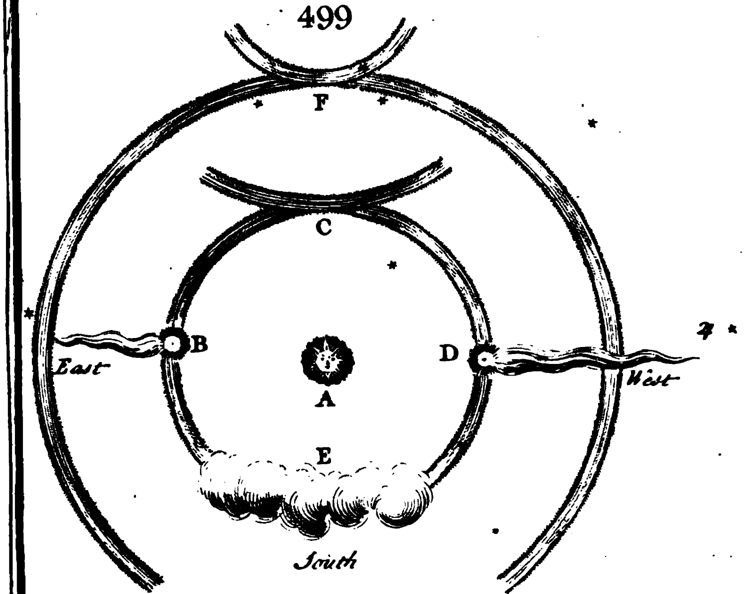
South

499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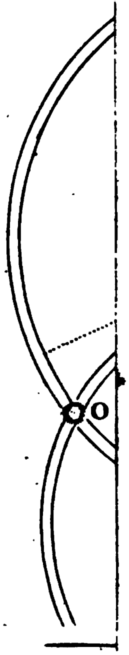
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covers nothing elſe but the particulars of the Roman Phænomenon ;) it ſhould follow that ſomething elſe muſt happen in the air: and yet I ſhall ſhew it to be nothing more than a certain poſition of thoſe cylinders, which has not yet been conſidered. But in the firſt place let us repreſent the phænomenon it ſelf, by *Hevelius*'s figure deſcribed in his own words, taken from the appendix to his *Mercurius in Sole viſus.* pag. 174.

Fig. 495.

On Sunday in the year of our Lord 1661. Feb. 20. New Style, a little before eleven o' clock, the ſun being towards the ſouth and the sky very clear, there appeared ſeven ſuns together in ſeveral circles, ſome white and ſome coloured, and theſe with very long tails, waving and pointing from the true ſun, together with certain white arches croſſing one another. 1. The true ſun at A, being about 25 degrees, high was ſurrounded almoſt entirely by a circle whoſe diameter was 45 degrees, and which was coloured like the rain-bow with purple, red and yellow; its under limb being ſcarce 2$\frac{1}{2}$ degrees above the horizon. 2. On each ſide of the ſun, at B and C towards the weſt and eaſt there appeared two mock-ſuns, coloured, eſpecially towards the ſun, with very long ſplendid tails, of a whitiſh colour and terminating in a point. 3. A far greater circle YXHVZ almoſt 90 degrees in diameter encompaſſed the ſun and the former leſſer circle GBIC, and extended it ſelf down to the horizon. It was very ſtrongly coloured in its upper part, but was ſomewhat duller and fainter on each ſide. 4. At the tops of theſe two circles at G and H were two inverted arches, whoſe common center lay in the zenith, and theſe were very bright, and beautifully coloured. The diameter of the lower arch QGR was 90 degr. and that of the upper one THS was 45 degr. In the middle of the lower arch at G, where it coincided with the circle BGC, there appeared another mock-ſun, but its light and colours were dull and faintiſh. 5. There appeared a circle BEFDC much bigger than the former, of an uniform whitiſh colour, parallel to the horizon at the diſtance of 25 degr. and 130 degr. in diameter; which aroſe as it were from the collateral mock-ſuns B and C, and paſſed through 3 other parhelia of an uniform whitiſh colour like ſilver; one at D almoſt 90 degr. from the true ſun towards the eaſt; another at E towards the weſt; and a third at F in the north, diametrically oppoſite to the true ſun; all of the ſame colour and brightneſs. There paſſed alſo two other white arches EN, DP of the greateſt circle of the ſphere, through the eaſtern and weſtern mock-ſuns E, D, and alſo through K the pole of the ecliptick; they went down to the horizon at N and P, croſſing the great white circle obliquely, ſo as to make a white croſs at each parhelion. So that ſeven ſuns appeared very plain at the ſame time; and if I could have ſeen the phænomenon ſooner from an eminency, I do not queſtion but I ſhould have found two more at H and I, which would have made nine in all. For there remained in thoſe places ſuch marks as made this ſuſpicion not improbable.

This moſt delightful and extraordinary ſight laſted from 30 minutes paſt 10 to 51 min. paſt 11; though it had not the ſame appearance all that while, but ſometimes one and ſometimes another. It appeared in the perfection of this deſcription at about eleven a clock, and then degenerated by degrees. The northern mock-ſun at F *vaniſhed firſt of all together with a part if its circle; the other parhelia with their arches laſted till ten minutes paſt eleven, then the eaſtern mock-ſun and after that the weſtern vaniſhed with both the croſſes. Soon after this the collateral parhelia* C, D *ſuffered ſeveral changes, ſometimes one was brighter than the other in light and colours, and ſometimes fainter and darker. For at 18 min. paſt 10, the eaſtern parhelion* C *vaniſhed while the weſtern parhelion at* B *remained very conſpicuous; and 24 min. paſt 11 the eaſtern one was very bright again and remained ſo while the weſtern one diſappeared at 40 min. paſt 11; although this weſtern one had almoſt always the longer tail. For the tip of it was frequently extended for 30 degr. and ſometimes 90, as far as the parhelion* E; *but the tail of the eaſtern one* C *was ſcarce above 20 degrees. At 30 min. paſt 11 the great vertical circle* YXHVZ *was deſtroyed; but the inverted arches* H *and* G *together with the collateral parhelia* B *and* C, *continued to the laſt.*

The ſcheme of this phænomenon is drawn in the ſame manner as the conſtellations are drawn upon an artificial globe, to be viewed by the eye on the outſide of it. For by this means every thing is repreſented much clearer and diſtincter. Nevertheleſs the place of the obſerver was nearly under the zenith within the circle parallel to the horizon; ſo that the true ſun appeared to him in the meridian, the mock-ſun F *in the north, and the other two at* D *and* E *on each hand. But if you deſire to have this extraordinary phænomenon repreſented a little plainer; upon an artificial globe whoſe pole is elevated to our altitude at* Dantzick, *with the center* A, *in the 2d deg. of Piſces, where the ſun then was, and with a ſemidiameter of 22½ degrees, deſcribe the circle* GBIC; *2. and then the circle* YXHVZ *with a radius of 45 degrees; 3. and with the ſame center and ſemidiameter of 90 degrees, draw the circle* NEKDP *through the two white mock-ſuns* E, D. *4, And with a ſemidiameter of 22½ the zenith being the center, draw the arch* THS; *5. and alſo the arch* QGR *with a radius of 90 degrees upon the ſame center; 6. and laſtly the circle* BEFDC *parallel to the horizon with a radius of 90 degrees. And the draught being finiſhed in this manner will appear very beautiful and harmonious; and, by the grace of God, may afford an occaſion of diſcovering the natural cauſes of all ſorts of parhelia and paraſelenæ.*

XXXVI. A ſolution of ſeveral particulars of this phænomenon.

562. In this phænomenon, as well as in that at *Rome*, we have the large white circle; the collateral parhelia *B*, *C* and the corona *BGCI*; all which are produced by cylinders in an upright poſture, as we have explained above; and likewiſe the parhelia *D*, *E*, with the arches *NE*, *PD* paſſing through them, for the reaſons given in the xxi and xxii ſections. And by what has been ſaid in xxiii and xxxiv ſections it appears why this circle *NEKDP*,

NEKDP, which has the ſun for its center, had a little opening at the top of it: and in the xxxii ſection it is explained how it happened that no back parhelia appeared. The diſtance of the two parhelia *E* and *D* from the ſun *A*, ought to have been within 88° 48′. For when the ſun was 25 degrees high the lateral parhelia could not appear at a greater diſtance from him[a]; and being at this limit, they ought to appear very faint and languid in compariſon to the two parhelia *B* and *C*. For this reaſon they appeared white as well as the arches *EN*, *DP* according to ſect. xxxi. The circle *ZHY* was produced either by round grains or by cylinders flying about confuſedly according to ſect. xx; or laſtly for another reaſon to be mentioned in the xli ſection.

XXXVII.
A great many cylinders lye horizontal.

563. But the chief thing to be inquired into, is the cauſe of the arches *THS*, *QGR* and of the parhelia *G* and *H* that appeared in them. For the parhelion *F*, diametrically oppoſite to the true ſun, ſhall be conſidered laſt of all. We obſerved above that the cylinders were ſupported in the air by vapours aſcending from below like a wind; which contributed both to their production and to their upright ſituation. Now beſides this upright poſture I find that moſt of the cylinders ought to affect an horizontal one; and it appears by experiments that they really muſt do ſo. For if a parcel of cylinders be ſuffered to deſcend with a gentle motion either through the air or through water; they will almoſt always fall down in an horizontal poſture; that is as ſoon as they have deſcended ſo far that their celerity cannot be farther increaſed. Now if the water or air through which theſe cylinders deſcend, be ſuppoſed to tend upwards with the ſame velocity as the cylinders tended downwards; the caſe comes to the ſame as that of little cylinders ſupported in the atmoſphere: ſo that no one can doubt but a great number of them lye horizontal. Indeed from the experiments above mentioned it might be inferred, that almoſt all of them ſhould be horizontal, and ſcarce any of them upright. But it muſt be conſidered that our experiments are but an imperfect imitation of what nature does in the atmoſphere; for the upright poſture is abundantly atteſted by thoſe effects which have been clearly demonſtrated to follow from it.

XXXVIII.
Hence follows thoſe arches that are convex towards the ſun.

564. I ſay then that theſe inverted and coloured arches, ſuch as *QGR*, *THS*, are produced by thoſe cylinders, whoſe axes are parallel to the plane of the horizon, though not to one another. To underſtand this matter, firſt let us ſuppoſe a vaſt number of cylinders lying promiſcuouſly, with their axes pointing towards the eaſt, ſouth-eaſt, ſouth, weſt, and all other points of the compaſs; and then let us conſider that the rays which produce theſe arches are tranſmitted through the cylinders by two refractions, in like manner as thoſe were which produced the collateral parhelia. Now ſuppoſing the ſun to be in the ſouth, for example, it will follow that thoſe parts of theſe arches, at *H* and *G*, which lye neareſt to the ſun, will be produced in thoſe cylinders whoſe axes point directly eaſtwards or weſtwards;

[a] Art. 543.

wards; that is in thoſe cylinders which receive the ſun's rays at right angles to their axes. For of all the cylinders in all poſitions, theſe that point eaſtward or weſtward are the neareſt to the ſun that can tranſmit his rays to the eye[a]. And for other cylinders to do the ſame thing in other poſitions, it is requiſite they ſhould lye farther and farther from the ſun; according as their axes point farther and farther from the eaſt or weſt. Becauſe the rays of the ſun are more and more elevated above the planes of their baſes; and conſequently produce the ſeveral parts of theſe inverted arches, which lye farther and farther from their middle parts at *G* and *H*. Having carefully conſidered what ſort of ſhapes theſe arches will have[b], I find this difference: *viz.* when the ſun is in the horizon the inverted arches, one of which touches the circle of 45 degrees and the other that of 90 degrees, are like the horns in the 496th figure. But the leſſer of the two horns ſhews the ſhape [of the arch[c]]; and the red colour is always next the ſun. The ſame figure ſhews alſo the ſaid arches when the height of the ſun is 10, 20, 30, 40, degrees. From which it appears that the middle of theſe horns are like pieces of circles, but towards their extremities they are bent the contrary way. And ſince the ſun ſhines more directly upon thoſe cylinders which produce the middle of the arches, and more obliquely upon thoſe which produce their extremities, it is no wonder that only the middle parts are viſible, as being brighter than the reſt.

XXXIX. The parhelia in the inverted arches explained. Fig. 495.

565. Now the parhelia which appear exactly in the middle of theſe arches, as at *G*, are nothing elſe but the brighteſt parts of them; and therefore never appear quite diſtinct nor much brighter than the adjoining parts of the arches. Accordingly *Hevelius* in all his obſervations of this kind, that is, of two parhelia and one paraſelenæ, has taken notice that they appeared ſomewhat dull, heavy and faint. And he was not ſure whether there was a parhelion at *H* or not. Now the reaſon why the middle part of the arch appears a little diſtinct from the reſt may be this. That though there be great plenty of horizontal cylinders in the air, yet they may be ſo ſhort as to have but little of a cylindrical ſurface, or perhaps may be like oblong ſpheroids. Beſides we find it perfectly agreeable to *Hevelius*'s obſervations, that the higher the ſun or moon and the corona are elevated above the horizon, the inverted arches appear the flatter. For in his obſervation of the paraſelenæ, in which the moon's altitude was 26 or 27 degrees, the arch over the firſt corona was part of a very large circle; as likewiſe in the phænomenon of 7 ſuns fig. 495. But in that of Apr. 6. and Decemb. 17. 1660, in which the ſun's altitude was but 12 or 13 degrees, the inverted arches were parts of much ſmaller circles. It muſt be confeſſed that the upper parts of theſe arches are repreſented much leſs by *Hevelius* than according to the following calculation. But the reaſon is, that arches ſo highly elevated above the horizon do neceſſarily deceive the ſight; ſo as to appear to be parts of much leſſer circles than they really are. For by what

Fig. 499 to 501.

[a] Art. 543.

[b] See Art. 567.

[c] minor autem duorum cornuum figuram exhibet.

what has been ſaid in the xxix ſection, it is evident that the ſame circle overhead appears but half as big as when it is near the horizon, which is true likewiſe of the parts of arches.

XL. Inverted arches below the corona's.

566. As to the arches which according to our hypotheſis ought to adhere to the loweſt part of the interior corona; in *Hevelius*'s two obſervations upon Mar. 30. 1660. fig. 499. and in that of the 7 ſuns, theſe arches might have been obſerved; and indeed they did appear in ſome meaſure, eſpecially in the laſt obſervation. For he ſays *there was ſome remains of a parhelion at* I fig. 495. but it appeared very faint for want of matter which ſeldom extends ſo far.

LXI. How to find the ſhapes of the inverted arches. Fig. 497.

567. The way to find the ſhapes of thoſe inverted arches is this. Upon a ſpherical ſurface *ABC* with the pole *B* deſcribe a great circle *ADC* repreſenting the horizon; then let *BED* be a vertical circle paſſing through the ſun at *E*; ſo that *DE* may be the ſun's altitude at the time of obſervation. Now to find the inverted arch which touches the leſſer of the two corona's; in the vertical *BED* take *EF* and *EG* each equal to 22 degrees, and the points *F*, *G* will be the higheſt and loweſt points of the corona, and likewiſe the middle points of the inverted arches which we are going to determine. To find any other points of them by means of the horizontal cylinders in various poſitions, let us imagine a great many of them to have their baſes parallel to a vertical circle *BM*, and let the circle *HEK* be parallel to it and paſs through the ſun. Now if we recollect what has been demonſtrated above concerning the effects of upright cylinders[a], and if we imagine the circle *BM* to be our horizon in whoſe center *N* the ſpectator is placed; and if we ſuppoſe the circle *HEK*, parallel to *BM*, to be the white circle paſſing through the ſun; the arch *HM* will be the ſun's altitude above the horizon *BM* and alſo his altitude above the planes of the baſes of the cylinders we are now concerned with; for we ſuppoſed their baſes parallel to the circle *BM*. Things being thus conceived we have nothing elſe to do but to determine the points *K*, *L* in the circle *HEK* where the lateral parhelia would appear; which is eaſy to be done by the table in the xvii ſection. For example let the arch *HM*, which is imagined to meaſure the ſun's altitude above the baſes of the cylinders, be 30 degrees; and the table ſhews that each of the arches *EK*, *EL* are 28° 48′; and one of the points *K*, *L* ſo determined, will be in the upper, and the other in the under inverted arch: and by the ſame means we may find as many points as are neceſſary to diſcover the ſhape and bending of the arches *FK*, *GL*. And by the ſame method we may determine the inverted arches that touch the higheſt and loweſt points of the larger corona. But before we diſmiſs the conſideration of this phænomenon, let us obſerve that the circle *ZHY*, fig. 495, might alſo be produced by theſe horizontal cylinders, although in the xxxvi ſection we aſcribed its origine to the cylinders in all manner of ſituations, or to the half thawed grains. For as we

[a] Art. 543.

have ſhewn in the xxi ſection how the upright cylinders produce a corona, ſo the horizontal ones may do the ſame thing; becauſe their extremities may be terminated by hemiſpheres or half ſpheroids. And it is very probable that this was the true cauſe of the circle *ZHY* in this phænomenon; becauſe we ſee it proceeds downwards on each ſide from the parhelion *H*; and we ſhall alſo obſerve the ſame thing in other parhelia. For this ſeems to intimate that the ſame horizontal cylinders produced both the circles we are ſpeaking of, and alſo the inverted arches.

LXII. *Scheiner's* obſervation in 1630 explained. Fig. 498.

568. Theſe things being underſtood we ſhall find no difficulty in explaining that other obſervation of *Scheiner*'s in 1630 in which there appeared ſix ſuns as deſcribed in the appendix. We need only obſerve that the parhelia *O* and *P* appeared in the interſections of the larger corona and of the white circle, becauſe this corona was produced by upright cylinders according to ſection xxi. Nevertheleſs ſince the higher parhelion *R* is alſo in this corona, the upper part of it may alſo be produced by horizontal cylinders in the manner abovementioned. There was no inverted arch obſerved to paſs through the parhelion *Q*, becauſe it might not be much extended on each ſide of *Q*; and therefore might not recede far enough from the corona *ZQβ* to appear diſtinct from it. For the ſun's altitude being 28½ degrees, this arch was bent upwards but very little; becauſe we have already obſerved it was very flat when the ſun's height was 27 degrees. The cauſe of the double circle that compoſed the inner corona has been mentioned in the xxii ſection.

LXIII. A ſolution of 3 other phænomena obſerved by *Hevelius*. Fig. 499 to 501.

569. The cauſes of three other phænomena in the appendix, obſerved by *Hevelius* the 30 Mar. the 6 Apr. and the 17 Decemb. are ſufficiently evident from what has been ſaid. Except that in the laſt of them there appeared a white croſs, made by two beams of light, one vertical and the other horizontal, that croſſed each other in the diſk of the moon. The reaſon is evident from thoſe vertical and horizontal cylinders which produce the paraſelenæ and circles paſſing through them. For the horizontal beam was nothing but a piece of the large white circle that uſually paſſes through the ſun, as explained in the xiii ſection. The reaſon why this horizontal beam does not always appear may be the want of matter, that is of upright cylinders; or it may be extinguiſhed by the ſuperior brightneſs of the ſun or moon, as we obſerved in the xiv ſection.

XLIV. The vertical beam explained. Fig. 501.

570. The other beam that was perpendicular to the horizon was produced by reflection of the moon's rays from the cylinders that lay horizontal; by which alſo theſe inverted arches were made, as explained above. Now ſince the ſides and axes of theſe cylinders are parallel to the plane of the horizon though not to one another; it follows when the ſun's altitude is ſo ſmall as in this obſervation, (for it was but 12 degrees), that a great multitude of thoſe that appear to be ſpread from the horizon up to a certain diſtance above the moon, will reflect the moon's rays to the eye as

if they came from this perpendicular beam: which will be the narroweſt and better defined at its edges next the moon than any where elſe; and the lower part between the moon and the horizon will be more perfect than the other part above the moon; where it ought to be dilated till it vaniſhes. Which ſeems to be agreeable to the phænomenon, ſo far as appears by the draught of it. In one thing indeed there is a difficulty in the theory; that is how an upright beam, cauſed by theſe reflections, could be ſo neatly terminated on each ſide as the figure ſeems to require; becauſe many of the horizontal cylinders that appear out of this beam may reflect rays to the eye; but the number of them is far leſs than the number of thoſe that lye in the beam. What has been ſaid of this bright beam may be demonſtrated by the 497 figure. For let *ADC* be the horizon, as before; *E* the moon whoſe altitude, meaſured by the vertical arch *ED*, is 12 degrees; *BM* another vertical ſuppoſed to be parallel to the baſes of a certain portion of the horizontal cylinders; *HEK* a leſſer circle parallel to *BM*; and of this portion of cylinders thoſe only that appear to lye in this circle *HEK* will reflect the moon's rays to the ſpectatot at *N*. In like manner every other portion of cylinders that have other particular directions, will have a lucid circle paſſing through the moon and extended down to the horizon. And as this circle *KEH*, in the parts next the moon at *E*, approaches very near to the vertical paſſing through the moon, ſo does a great number of other circles; ſo that by all of them taken together there is formed a bright ſort of a column paſſing through the moon's vertical *BED*. And this column will be ſo much the brighter and more conſpicuous as the moon is lower. Becauſe more of theſe lucid circles are crowded into a narrower ſpace, and are produced by a greater number of cylinders when the moon is low than when ſhe is higher elevated.

Fig. 497.

XLV. Another meteor of the ſame ſort explained.

571. This was alſo the cauſe of another meteor obſerved at *Caſſel* Jan. 2. 1586. by *Chriſtopher Rothman*; which he relates in this manner, in his deſcription of a comet in the year 1586. *The ſky being very clear in the eaſt, juſt before ſun-riſe there appeared an upright column exactly ſituated in a vertical circle. Its breadth was every where equal to the ſun's diameter; and it looked as if ſome village was on fire beyond the mountains. For it appeared like a column of flame, excepting that its thickneſs was every where the ſame. Soon after in the ſame column there aroſe an image of the ſun exactly reſembling the true ſun. There was ſcarce one digit of this image under the horizon, when the true ſun began to riſe in the ſame column, which was followed in like manner by another image. The column with its three ſuns touching one another continued always upright or in a vertical circle, as appeared by the plummet of a quadrant. Theſe ſuns had all the ſame appearance except that the true ſun in the middle was brighter than the reſt. This appearance of the column paſſing through three ſuns laſted al-*

moſt a quarter of an hour, till they were covered by a black cloud deſcending from above.

The reflection of the ſun's rays from the horizontal cylinders was the cauſe of this appearance; unleſs upon account of the images of the ſun, it be more reaſonable to ſuppoſe there was a large quantity of upright cylinders in the atmoſphere, whoſe extremities were bounded by flat ſtars, ſuch as *Des Cartes* has obſerved to fall down upon the ground[a]. For theſe a Fig. 504. cylinders ſwinging ſeparate from one another (as I ſuppoſe) and being ſupported by a current of riſing vapours, ought to continue ſuſpended b Art. 538. for the moſt part in that upright poſture in which they were generated[b]; not exactly perhaps, but ſo as to decline but a little from it. Theſe flat ſtars are therefore ſo many ſmall planes, from which the rays of the ſun are reflected in ſuch a manner, that the ſuperior part of the column with its parhelion, may be produced by reflection from the inferior baſes of the cylinders; and the inferior part of the column with its parhelion from the ſuperior baſes. I obſerved that theſe cylinders might not be exactly upright but might incline a little, perhaps a degree or two; for if their baſes were exactly plane and parallel to the horizon, they could not reflect any of the ſun's rays to the eye.

The reſt is wanting in the original.

AN APPENDIX.

Containing the Obſervations quoted above, and a demonſtration of the Tables.

I. SCHEINER'S *Obſervation of Corona's and Parhelia.* A. D. 1630.

Taken from *Gaſſendus's* Works Tom. 6. p. 401. Fig. 498.

572. THE diameter of the corona *MQN* next to the ſun was about 45 degrees; and that of the remoter corona *ORP* was about 95°. 20′; they were coloured like the primary rain-bow, but the red was next the ſun, and the other colours in the uſual order. The breadths of all the arches were equal to one another, and about a third part leſs than the diameter of the ſun, as repreſented in the ſcheme. Though I cannot ſay but the whitiſh circle *OGP* parallel to the horizon, was rather broader than the reſt. The two parhelia *M*, *N* were lively enough; but the other two at *O* and *P* were not ſo briſk; *M* and *N* had a purple redneſs next the ſun; and were white in the oppoſite parts; *O* and *P* were all over white. They all differed in their durations. For *P*, which ſhone but ſeldom and but faintly, vaniſhed firſt of all, being covered by a collection of pretty thick clouds. The parhelion *O* continued conſtant for a great while though it was but faint. The two lateral parhelia *M*, *N* were ſeen conſtantly for three hours together: *M* was in a languiſhing ſtate and dyed firſt, after ſeve-

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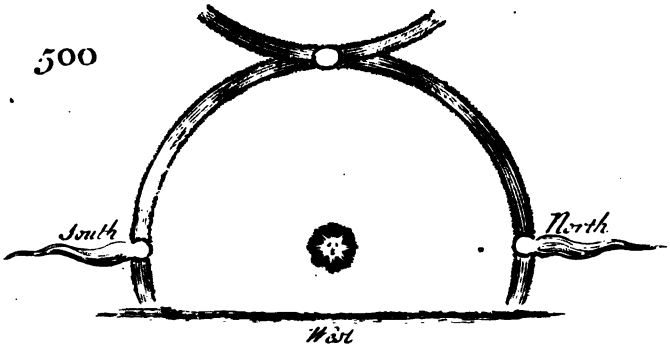
500



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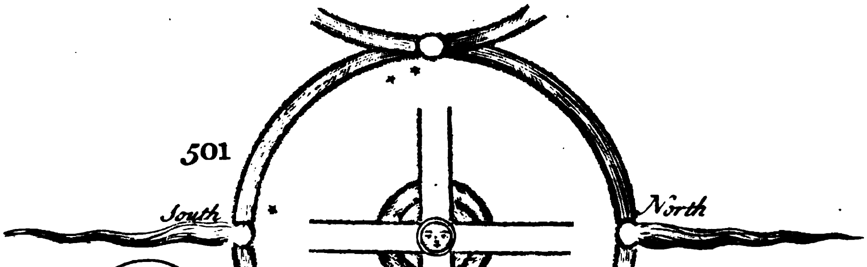

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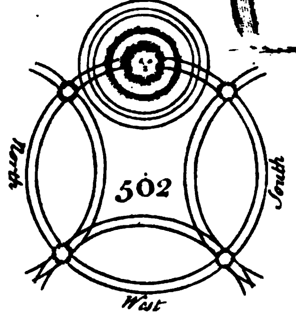

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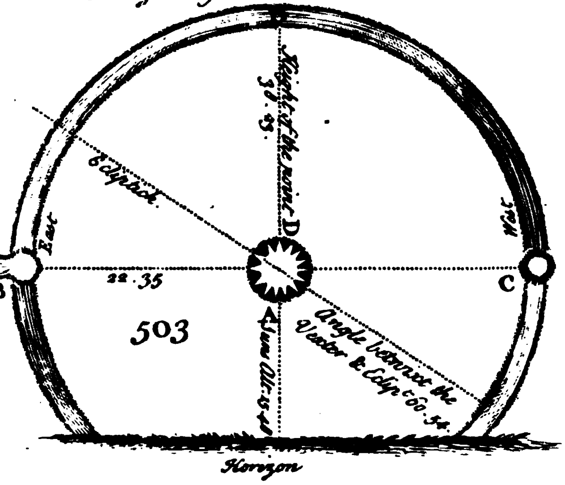
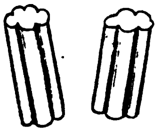
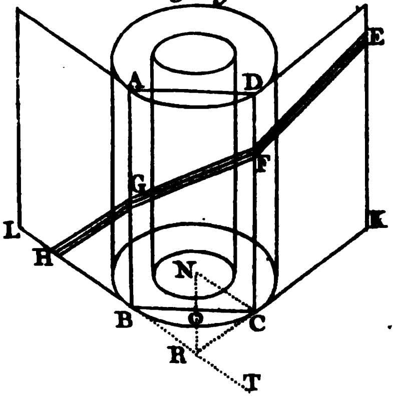

Height of the point D 38.25
Ecliptick
East
West
27.0
F
E
13.10
B
22.35
C
A
Sun Alt. 15.40
Angle betwixt the Vertex & Eclip. 60.34
Horizon

504



509



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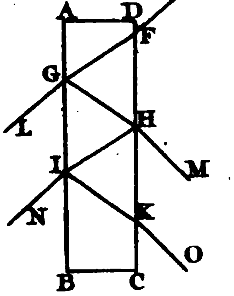
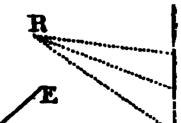




ſeveral ſtruggles; but *N* continued an hour after it at leaſt. Though I did not ſee the laſt end of it, yet I was ſure it was the only one that accompanied the true ſun for a long time, having eſcaped thoſe clouds and vapours which extinguiſhed the reſt. However it vaniſhed at laſt, upon the fall of ſome ſmall ſhowers. This phænomenon was obſerved to laſt 4 hours and a half at leaſt; and ſince it appeared in perfection when I firſt ſaw it, I am perſwaded its whole duration might be above five hours.

The parhelia *Q*, *R* were ſituated in a vertical plane paſſing through the eye at *F* and the ſun at *G*, in which vertical the arches *CRH*, *ORP* either croſſed or touched one another. Theſe parhelia were ſometimes brighter, ſometimes fainter than the reſt; but were not ſo perfect in their ſhape and whitiſh colour. They varied their magnitudes and colours according to the different temperature of the ſun's light at *G* and the matter that received it at *Q* and *R*; and therefore their light and colours were almoſt always fluctuating and continued as it were in a perpetual conflict. I took particular notice that they appeared almoſt the firſt and the laſt of all the parhelia excepting that at *N*.

The altitude of *Q* above the horizon, in the morning at the beginning of the obſervation, was 49° 40′; that of *R* was 76° 10′; that of the true ſun was 28° 30′. Hence the height of *Q* above the ſun was 21° 10′; and the height of *R* above the ſun was 47° 40′. There was a north wind at the beginning of theſe obſervations, but by degrees it changed to the eaſt and at laſt to the ſouth; yet it brought no very great nor laſting rains. For near a fortnight after the ſky looked always vapouriſh; and every day before dinner the ſun endeavoured to create new ſuns, but in vain, either for want of matter or of a due diſpoſition. For in the vertical circle I ſaw plainly ſome ſketches of parhelia for a long time. I ſaw alſo very manifeſt reciprocations of the lateral parhelia. The iris *ORP* ſeems to have been a portion of a ſingle circle concentrick to the ſun, but towards *κ* and *θ* it did not quite touch the horizon *AB*; and the lengths of the arches *Oκ*, *Pθ* were variable. The arches *ZQα*, *βQγ*, *δεζ*, that immediately ſurrounded the ſun, ſeemed to the eye to compoſe a ſingle circumference, but it was confuſed, and had unequal breadths; nor did it conſtantly continue like it ſelf, but was perpetually fluctuating. But in reality it conſiſted of the arches expreſſed in the ſcheme, as I accurately obſerved for that very purpoſe. The horns *HRC* ſeemed to be a portion of a ſmaller circle touching the greater *ORP* in a contrary poſition in a common knot at *R*. The arches *ZQα*, *βQγ* cut each other in a knot at *Q* and there they formed a parhelion. The parhelia *N*, *M* ſprung out from the common interſections *M*, *N* of the Iris *δεζ* and of the whitiſh circle *ONMP*. The north part of the ſky was clearer than the ſouth, which being overcaſt with ſlender vapours afforded more matter for this appearance.

II. HEVELIUS's *Obſervation of Paraſelenæ, ſeen at* Dantzick A.D. 1660. Mar. 30.

Fig. 499. 573. In the beginning at one a clock in the morning the moon *A* was ſurrounded by an entire whitiſh circle *BCDE*; in which there were two mock-moons at *B* and *D*, one at each ſide of the moon, conſiſting of various colours and ſhooting out very long and whitiſh beams by fits. That on the left hand extended its tail towards the thigh of Serpentarius, the other on the right extended its tail towards Jupiter, as repreſented in the figure. Afterwards, at two a clock, a larger circle ſurrounded the leſſer and reached down to the horizon. The tops of both theſe circles were touched by coloured arches like inverted rain-bows. The inferior arch at *C* was a portion of a larger circle, and the ſuperior a portion of a leſſer. This extraordinary ſight laſted near three hours; the outward great circle vaniſhed firſt of all, then the larger inverted arch at *C*, and preſently the leſſer; and laſt of all the inner circle *BCDE* diſappeared. The diameter of this inner circle and alſo of the ſuperior arch was 45 degrees; that of the exterior circle and inferior arch was 90 degrees.

III. HEVELIUS's *Obſervation of Parhelia* A. D. 1660. Apr. 6.

Fig. 500. 574. At half an hour paſt 5 in the evening while the ſun was deſcending towards the horizon, he was crowned with arches of circles of various colours like the rain-bow. In the corona on oppoſite ſides of the ſun there were two parhelia variouſly coloured, with pretty long and whitiſh tails pointing from the ſun. Near the zenith, where the corona was a little faint and imperfect, there ſhone out another inverted arch, having a third parhelion in the middle of it, which appeared ſomewhat obſcure. This phænomenon laſted half an hour, till ſun-ſet, the ſky being very clear. The inverted arch and the upper parhelion diſappeared firſt; and then the parhelion on the left hand; but the third parhelion ſet with the true ſun. The diameter of the corona round about the ſun was about 45 degrees, as I gueſſed by my eye.

IV. HEVELIUS's *Obſervation of Paraſelenæ ſeen at* Dantzick A. D. 1660. Decemb. 17.

Fig. 501. 575. On the firſt day after the full moon at 30 minutes paſt 6 in the morning, the moon being 12 degrees high, I ſaw the moon in the weſt with 3 mock-moons about her, in this manner. The air being very clear at firſt I obſerved the moon ſurrounded with a double corona, [near her body as the figure ſeems to repreſent,] tinged with very bright and beautiful colours. On each ſide of the moon there was two arches of a large circle about 45 degrees in diameter, which were alſo coloured like the rainbow, and exended down to the horizon. In which were two mock-moons with very long white tails. That on the left hand was near Procyon with a ſhort tail; the other on the right hand had a longer tail. In the upper part where theſe collateral arches concurred, there was another arch inverted and variouſly coloured, with a third mock-moon in the middle of it, and ſomewhat duller than the other two. Moreover what was very extraordinary, there paſſed a large, white, rectangular croſs through the diſk of the moon; whoſe lower part reached down to the horizon; but on each ſide it did not quite touch the corona; as appears by the figure. It was ſo very bright and ſtrong that it ſhone diſtinctly and clearly till ſun-riſe. But the mock-moons diſappeared a little before.

V. *An Obſervation taken from* MATTHEW PARIS's *Hiſtory.*

576. A wonderful ſight was ſeen in England A.D 1233. Apr. 8. in the 17th year of the reign of HENRY III. and laſted from ſun-riſe till noon. —— At the ſame time on the 8th of April about one a clock, in the borders of *Herefordſhire* and *Worceſterſhire*, beſides the true ſun there appeared in the ſky four mock-ſuns of a red colour; alſo a certain large circle of the colour of cryſtal, about two foot broad, which encompaſſed all England as it were. There went out ſemicircles from the ſide of it, in whoſe interſections the four mock-ſuns were ſituated; the true ſun being in the eaſt and the air very clear. And becauſe this monſtrous prodigy cannot be deſcribed by words I have repreſented it by a ſcheme, that ſhews immediately how the heavens were circled. The appearance was painted in this manner by many people, for the wonderful novelty of it. Fig. 502.

VI. *A true delineation of a parhelion ſeen at* Leyden A.D. 1653. Jan. 14. *between one and two a clock in the afternoon; and obſerved in the Academical Obſervatory by* SAMUEL CHAR. KECHELIUS *a* Hollenſtein.

577. *BDC* was a white circle almoſt 35 min. broad; the altitude of its higheſt point *D* was 38° 23′. Its center was in the ſun, whoſe height was 15° 48′, that is at 36 minutes paſt one; his Azimuth being 23° 40′ towards the weſt, and the angle made by his vertical circle and the ecliptick 60° 54′. The mock-ſuns *B*, *C* were oblong and unequal, at the diſtance of 22° 35′ on each ſide of the ſun, and had the ſame altitudes as the ſun. The weſtern parhelion at *C* was the fainter of the two and changed from yellow to white and diſappeared firſt; the eaſtern one at *B* was brighter, with a lucid arch ſhooting from the ſun, and was coloured with purple, red and yellow; the ſhape of its tail *BF* was conical, 27 degrees long, the parhelion being the baſe of the cone; the part *BE*, 13° 10′ long, conſiſted of bright yellow and red light, the other part *EF* being whitiſh; which vaniſhed Fig. 503.

vaniſhed before the parhelion did. It appeared for half an hour and laſted one quarter longer than *C.* And the corona diſappeared a little after.

VII. HUGENIUS's *Obſervation* 13 May 1652.

578. I obſerved a circle about the ſun in its center; its diameter was about 46 degrees; and its breadth the ſame as that of the common rainbow; it had alſo the ſame colours, though very weak and ſcarce diſcernable, but in a contrary order, the red being next the ſun; and the blue very dilute and whitiſh. All the ſpace within the circle was poſſeſſed by a duller vapour than the reſt of the air; of ſuch a texture as to obſcure the ſky with a ſort of a continued cloud, but ſo thin that the blue ſky-colour appeared through it. The wind blew very gently from the north.

A DEMONSTRATION *of the* CONSTRUCTION *of the* TABLES, *and of ſome other Mathematical Propoſitions relating to the foregoing Diſſertation.*

PROPOSITION I.

579. *A ray of the ſun reflected from any vertical plane makes the ſame angle with the horizon as the incident ray does; or, the ſun ſeen by reflection from any vertical plane appears of the ſame height as when ſeen directly.*

Fig. 505. Let EF and EG be an incident and reflected ray from the vertical plane LI; and let a plane $MFEGN$ paſſing through the incident and reflected rays, cut the reflecting plane LI at right angles in the line MN. From any point G in the reflected ray draw GN perpendicular to MN and produce it through the plane LI till it meets the incident ray FE, produced through it in G. Laſtly let $PGNg$ repreſent an horizontal plane cutting the vertical line OEP in P. Now ſince the angles MEF and NEG or NEg, which are the complements of the angles of incidence and reflection, are equal; and ſince the line GNg is perpendicular to and biſected by the horizontal line PN, the triangles NPG, NPg are alſo equal. Hence alſo the triangles PEG, PEg being both right angled at P, are alſo equal; and ſo the angles PgE, PGE, that is the altitudes of the incident and reflected rays are alſo equal. *Q. E. D.*

580. *Corol.* 1. Hence the vertical planes PEg, PEG paſſing through the incident and reflected rays, are equally inclined to the reflecting plane LI; their inclination to it being meaſured by the equal angles NPg, NPG.

581. *Corol.* 2. And if the ſun be ſeen by rays ſucceſſively reflected from the concave or convex ſides of any number of cylinders, whoſe axes are

vertical, he will appear of the ſame altitude, as if ſeen by direct rays. For if the plane *LI* be conceived to touch the ſide of the cylinder along the vertical line *OEP*, the reflection at *E* from the ſide of the cylinder will be juſt the ſame as from the tangent plane[a].

582. *Corol.* 3. Hence alſo if the ſun be ſeen by rays ſucceſſively reflected from any number of vertical planes, he will appear of the ſame height as if ſeen by direct rays.

PROPOSITION II.

583. *If a ray of the ſun be refracted through a vertical plane in any poſition, the ſine of the altitude of the incident ray above the horizon, will be to the ſine of the altitude of the refracted ray above the horizon, in the given ratio of the ſine of incidence to the ſine of refraction.*

For let *MN* be the refracting vertical plane; *EF* an incident and *FG* the refracted ray; *EF* being ſuppoſed on the foreſide and *FG* on the backſide of the plane *MN*; and let a plane *EIFGQ* paſſing through the rays *EF*, *FG*, cut the plane *MN* at right angles in the line *IQ*. This being the plane in which the refraction is made, take *FE* and *FG* of any equal lengths, and drawing *EI* and *GQ* perpendiculars to the plane *MN*, the angle *IEF* will be equal to the angle of incidence, and *FGQ* to the angle of refraction; whoſe ſines are *FI*, *FQ*. Draw the vertical line *DFC*, and let two horizontal planes, paſſing through the horizontal lines *IE*, *GQ*, be repreſented by the triangles *DEI*, *CGQ*; and the angles *FDI*, *FDE*, and alſo *FCQ*, *FCG* will be right angles; and therefore the triangles *FDI*, *FCQ* will be ſimilar. Therefore *FI* is to *FQ* as *FD* to *FC*. That is, the ſine of incidence is to the ſine of refraction, as the ſine of the angle *FED*, the altitude of the incident ray, to the ſine of *FGC*, the altitude of the refracted ray. For *FE* and *FG* were put equal, and *FDE*, *FCG* are vertical planes. *Q. E. D.* Fig. 506.

584. *Corol.* Hence the apparent altitude of the ſun ſeen by refraction through any vertical plane, is always the ſame as it would be when the refracting plane is ſo ſituated that the plane of incidence is alſo a vertical plane; in which caſe the ſines of the altitudes of the incident and of the refracted ray, are the very ſines of incidence and refraction.

PROPOSITION III.

585. *In any given altitude of the ſun, the ſines of the angles, which the vertical planes drawn through the incident and the refracted ray, do make with a vertical plane perpendicular to the refracting vertical plane, are in a given ratio; compounded of the ratio of the ſines of the altitudes of the incident and of the refracted ray directly and of their coſines inverſely.*

[a] Art. 19.

Fig. 506. Things remaining as they were, produce the incident ray EF along the plane of refraction FGQ till it cus GQ in L; and the line joining the points L, C is the horizontal ſection of the vertical plane FCL paſſing through the incident ray EF; and CG is the horizontal ſection of the vertical plane FCG paſſing through the refracted ray FG; and laſtly GQ is the horizontal ſection of a third vertical plane conceived to ſtand perpendicular to the refracting plane MN. Therefore the inclinations of the two former planes to the latter, are meaſured by the angles CLQ, CGQ. * Art. 221. Now in the triangle CLG, the ſine of the angle CLG or of CLQ, is to the ſine of CGQ, as CG, to CL*, or as $DF\times CG$ to $(DF\times CL=)\ CF\times DE$; becauſe DF is to CF as DE to CL. Now the ratio of $DF\times CG$ to $CF\times DE$, is compounded of the ratio of DF, the ſine of the angle DEF which meaſures the altitude of the incident ray, to CF, the ſine of the angle CGF which meaſures the altitude of the refracted ray; and of the ratio of CG, the coſine of the latter angle, to DE, the coſine of the former. Q. E. D.

586. *Corol.* Hence putting I to R for the ratio of the ſines of incidence and refraction, the radius FE or $FG=r$, the ſine of the ſun's altitude $FD=s$; we have the coſine $DE=\sqrt{rr-ss}$, and the ſine of the altitude of * Art. 583. the refracted ray FG, that is $FC=\frac{R}{I}s$*, and its coſine $CG=\sqrt{rr-\frac{RR}{II}ss}$. Hence putting P to Q for the ratio of the ſines of the angles CLQ, CGQ, by this propoſition we have $\frac{P}{Q}=\frac{I}{R}\times\frac{\sqrt{rr-\frac{RR}{II}ss}}{\sqrt{rr-ss}}$; which will be of uſe hereafter. To compute this ratio, when s is the ſine of 25 degrees and I to R as 250 to 187 (which is the ratio that *Hugenius* uſes in the conſtruction of his Tables) we have

Log. ſin. 25° or the Logarithmick ſine s - - - - -	9. 62594
Log. $\frac{I}{R}$ - - - - - - - - - - - - - -	0. 12610
Hence Log. $\frac{R}{I}s$ - - - - - - - - - - - -	9. 49984
The Log. ſine complement of this laſt ſine, is the log. $\sqrt{rr-\frac{RR}{II}ss}$	9. 97713
Log. ſine compl. 25° is the log. $\sqrt{rr-ss}$ - - - - -	9. 95727
Hence Log. $\frac{\sqrt{rr-\frac{RR}{II}ss}}{\sqrt{rr-ss}}$ - - - - - - - - - -	0. 01986
Log. $\frac{I}{R}$ was - - - - - - - - - - - - -	0. 12610
Hence Log. $\frac{P}{Q}=\frac{I}{R}\times\frac{\sqrt{rr-\frac{RR}{II}ss}}{\sqrt{rr-ss}}$ is - - - - - -	0. 14596

PROPOSITION IV.

Fig. 507. 587. *If a ray of the ſun,* EF, *falling obliquely upon the outſide of a pellucid upright cylinder,* AC, *whoſe baſes are parallel to the horizon, be refracted into it at* F, *and be reflected within it at* G, *any number of times or no times at all, before it emerges out again by refraction at* H; *the ſun ſeen by any emergent ray whatever, as* HK, *will appear of the ſame altitude as if he was ſeen by direct rays.*

Through the points of incidence, reflection and refraction, *F, G, H,* draw the vertical lines *PC, AQ, BD*; and ſuppoſing vertical planes to touch the cylinder in every one of theſe lines, the refractions and reflections at theſe planes will be juſt the ſame as if they were made at the cylindrical ſurface. Put *I* to *R* for the ratio of refraction, and the ratio of the coſines of the angles *PFE, CFG,* (that is of the ſines of the altitudes of the incident and refracted rays) is the ſame as *I* to *R* *; and the ratio of the coſine of *CFG* or *AGF* or *QGH* or *GHB* (which are all equal[a]) is to the coſine of *KHD* as *R* to *I* *; therefore by compounding theſe ratios the coſine of *PFE* is to the coſine of *KHD* in a ratio of equality, and conſequently the angles themſelves and their complements are equal. *Q. E. D.*

* Art. 583.

[a] Art. 581.

* Art. 583.

ANOTHER DEMONSTRATION.

Fig. 508. 588. Let the parallelogram *ABCD* be a vertical ſection through the axis of the cylinder and through an incident ray *EF*, and ſince the refracted and reflected rays, *FG, GH, HI, IK,* &c. make equal angles with the parallel lines *AB, CD*; it is evident that the incident ray *EF* and the emergent ray *GL* or *HM* or *IN* or *KO*, &c. will be all inclined to the vertical lines in the ſame angles. And all other rays which fall obliquely on the cylinder, will alſo, after refractions and reflections, be inclined to their vertical lines in the ſame angles[b]. *Q. E. D.*

[b] Art. 581. 584.

PROPOSITION V.

589. *When the rays of the ſun are ſo refracted through the ſides of an erect cylinder, as to touch the ſide of a leſſer cylinder upon the ſame axis, it is propoſed to find the inclination of the vertical plane of incident rays, to the vertical plane of emergent rays, when the ſun has any given altitude and the ratio of the diameters of the cylinders is given.*

Fig. 509. Let three vertical planes paſſing through the incident, refracted, and emergent rays *EF, FG, GH,* be repreſented by the parallelograms *DCK, DCBA, ABL,* and let them cut the horizontal plane of the baſe of the cylinders in *CK, CB, CL*. From *N* the center of the baſe draw *NO* perpendicular to the chord *BC* and join *NC*; and having produced *CK, BL*

till they meet in R, the angle KRL or its ſupplement KRT will be the meaſure of the inclination of the planes DCK, ABL. And ſince NC is the horizontal ſection of a vertical plane NCF perpendicular to the refracting ſurface at C or F, or to a plane that touches the cylinder in the line CF; by art. 585 the ſine of the angle NCR is to the ſine of the angle NCO, in a ratio compounded of the ſines of the altitudes of the incident and of the refracted rays directly and of their coſines inverſely; and conſequently is given by art. 586. But NO the ſine of the angle NCO is given from the given proportion of NC to NO, that is of the ſemidiameters of the cylinders; therefore the angles NCO, NCR are both given, and alſo their difference BCR or CBR, which are equal; and the ſum of theſe angles equals the external angle KRT. *Q. E. I.*

590. For example, to ſhew the conſtruction of the Table in the xvii ſection [a]; when the ſun's altitude was 25° we computed the Log. of P to Q or of the given ratio of the ſines of the angles NCR, NCO, and found it 0. 14596 [b]. Now putting the ratio of the ſemidiameters of the cylinders, that is NC to NO as 1000 to 473, the angle NCO will be 28° 14′, whoſe Log. ſine is 9. 67492; to which adding 0. 14596, the ſum is 9. 82088, which is the Log. ſine of 41° 27′ = ang. NCR. Hence the difference of theſe angles is 13° 13′ = ang. OCR, whoſe double 26° 26′ = ang. KRT, the diſtance of the lateral parhelion from the ſun.

[a] Art. 553.

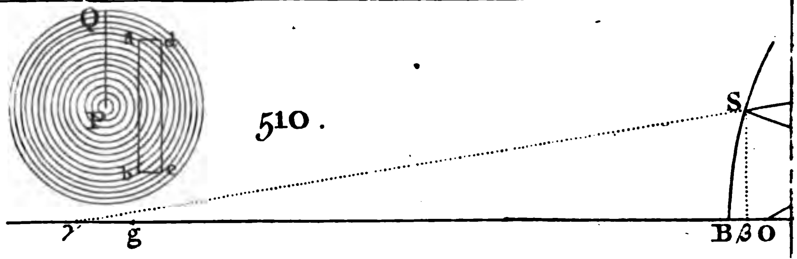
[b] Art. 586.

591. To find the greateſt quantity of the angle KRT, let us ſuppoſe the plane KCD to touch the cylinder, or the angle NCR to be a right one, whoſe logarithmick ſine is 10. 00000. Hence ſubſtracting 0. 14596 the log. of the ratio of the ſines of NCR, NCO, there remains 9. 85404 for the logarithmick ſine of the angle $NCO = 45° 36'$. Which being taken from $NCR = 90°$ gives $RCO = 44° 24'$, whoſe double $88° 48' = KRT$, the greateſt diſtance of the lateral parhelion from the ſun when he is 25° high. In this caſe therefore $NO : NC :: 714 : 1000$.

PROPOSITION VI.

592. *When the rays of the ſun are refracted through the ſide of an erect cylinder ſo as to touch the ſide of an inner concentrick cylinder of a given diameter, and are reflected at the ſurface of the outward cylinder any number of times before they emerge out again by refraction; it is propoſed to find the inclination of the vertical plane of the incident rays to the vertical plane of the emergent rays.*

Fig. 507. *Caſe* 1. Let EF, FG, GH, HK, be the courſe of the incident, refracted, reflected and emergent rays, and let as many vertical planes paſſing through them cut the horizontal plane of the baſe in LC, CQ, QD, DM reſpectively. Through N the center of the baſe produce NQ, and let it meet LC produced in R. Now from the given ſemidiameter NO of the inner cylinder, and the given ratio of the ſines of the angles NCR, NCO, their



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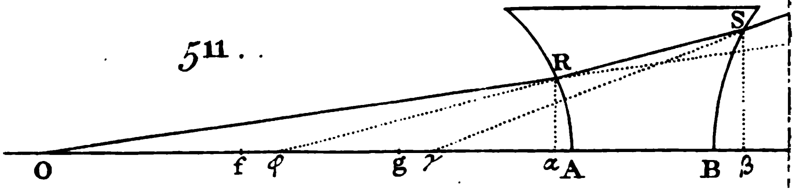

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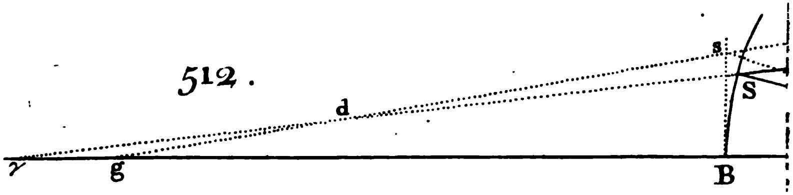

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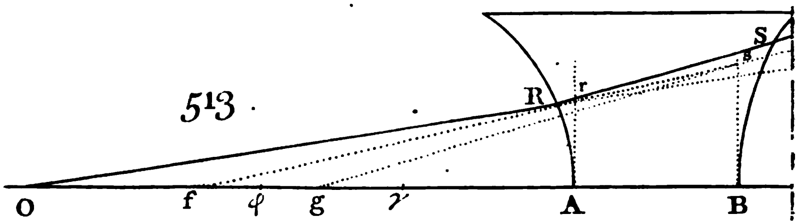

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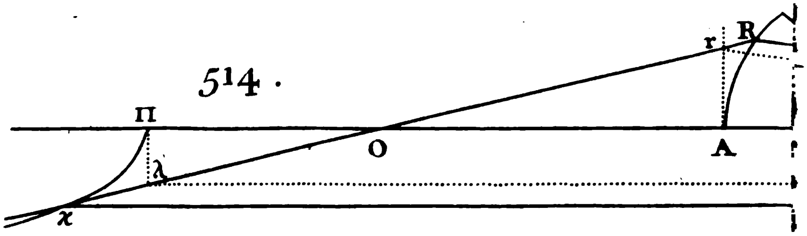

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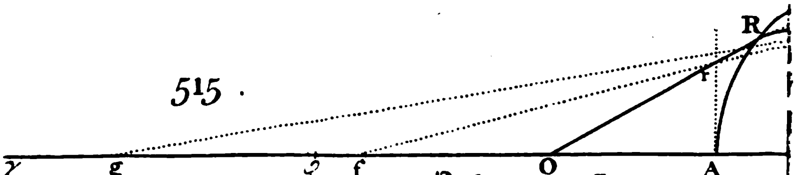

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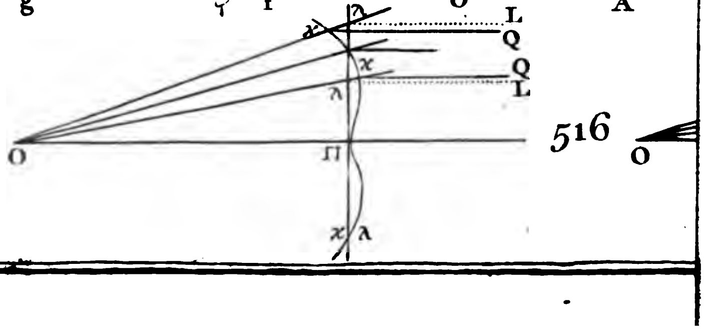

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their difference OCR will be given, juſt as in the foregoing propoſition; which being taken from NCQ or NQC gives NRC, which is half the angle LRM. For ſuppoſing the ray to be reflected at G by a plane touching the cylinder along GQ, this tangent plane and conſequently the plane GQN perpendicular to it, is equally inclined to the planes GQC, GQD cor. 1. prop. 1. that is, the angles NQC, NQD and conſequently NDQ, NCQ are all equal. Hence it follows, that the angle NDR is equal to NCR. For if the emergent ray be ſuppoſed to go backwards along KH, ſince by prop. 4. the altitudes of the incident and refracted rays at H are equal to thoſe at F, by propoſition 3 the ratio of the ſines of the angles NDR, NDQ will alſo be the ſame as that of the ſines of NCR, NCQ. Therefore as the angle NDQ is equal to NCQ, ſo NDR is equal to NCR and conſequently NRD to NRC. Q. E. J.

593. *Caſe* 2. Now whatever number of reflections be made before the ray emerges again by refraction, ſince the chord CQ was proved equal to QD, for the ſame reaſons QD will equal the chord lying under the next reflected ray, and ſo on; and ſince the ſines of the angles NCR, NCQ are in a given ratio *, they may be conſidered as the ſines of incidence and refraction of an horizontal ray LC; and therefore the angle under theſe incident and emergent rays, or under the planes that ſtand erect upon them, will be given in all caſes by art. 488. *Q. E. J.* * Art. 585.

PROPOSITION VII.

594. *The ſun ſhining upon an erect cylinder whoſe axis is vertical, to find the greateſt angle which a vertical plane of emergent rays, after any given number of reflections, makes with a vertical plane of the incident rays.* Fig. 507.

Suppoſing the conſtruction and demonſtration of the laſt propoſition, ſince the ſines of the angles of incidence and refraction of the vertical planes of incident and refracted rays, that is of the angles NCR, NCQ are in a given ratio * when the altitude of the ſun is given; by ſubſtituting the terms of this ratio inſtead of the ratio of the ſines of incidence and refraction, this problem is ſolved by the ſame rules as were given for the rain-bow *. Thus putting m for the number of reflections increaſed by an unite, P to Q for the ratio abovementioned, the ſine of the angle NCR will be $\frac{NC}{Q}\sqrt{\frac{mmQQ-PP}{mm-1}}$, and the ſine of NCQ will be $\frac{NC}{P}\sqrt{\frac{mmQQ-PP}{mm-1}}$. For in Fig. 470 we had $BCq : BFq :: mmRR-RR : II-RR$, and disjointly $BCq : CFq :: mmRR-RR : mmRR-II$*. Whence $CFq = \frac{BCq}{RR}\times\frac{mmRR-II}{mm-1}$. Hence the angle required is given by art. 491. *Q. E. J.* * Art. 585. * Art. 492. * Euc. I. 47.

595. *Corol.* When there is but one reflection, $m=2$; and therefore putting $NC=r$, the ſine of $NCR=\frac{r}{Q}\sqrt{\frac{4QQ-PP}{3}}=\frac{2}{\sqrt{3}}\sqrt{rr-\frac{PP}{4QQ}rr}$, which is an eaſier form for the following calculation.

596. To ſhew the conſtruction of the Table in the xxviii ſection. For example, when the ſun's height was 25° we found the Log. $\frac{P}{Q}$* 0. 14596

The Log. of r, or of the whole ſine, is - - - -	10. 00000
Whence Log. of $\frac{P}{Q}r$ is - - - - - - - - - - - -	10. 14596
Log. of the number 2 is - - - - - - - - - - - - -	0. 30103
Hence Log. $\frac{P}{2Q}r$ is - - - - - - - - - - - - -	9. 84493
The ſine compl. of which log. or the log. of $\sqrt{rr-\frac{PP}{4QQ}rr}$ is	9. 85398
Log. $\frac{2}{\sqrt{3}}$ is - - - - - - - - - - - - - - - - - -	0. 06247
The Log. of $\frac{2}{\sqrt{3}}\sqrt{rr-\frac{PP}{4QQ}rr}$ is - - - - - - - -	9. 91645
The angle NCR anſwering to this ſine, is 55° 35′	
Log. $\frac{P}{Q}$ was - - - - - - - - - - - - - - - -	0. 14596
Remains the log. of the ſine of NCO - - - - - -	9. 77049

The angle NCO anſwering to that ſine, is 36° 07′. Hence their difference $QCR = 19° 28'$, which being taken from NQC or $NCO = 36° 7'$, gives $QRC = 16° 39'$, whoſe double $LRM = 33° 18'$, the diſtance of either of the back parhelia from the vertical circle oppoſite to the ſun, when his altitude is 25 degrees.

597. Hence it appears that if the ſemidiameter of the inward cylinder of ſnow, be greater than $\frac{590}{1000}$ of the ſemidiameter of the whole cylinder, it will hinder the appearance of the back parhelia, by ſtopping the refracted rays when the ſun is 25° high; becauſe in this caſe NO, the ſine of the angle $NCO = 36° 07'$, is almoſt $\frac{590}{1000}$ of NC.

CHAPTER XII.

To determine the apparent ſhapes, poſitions, magnitudes and diſtances of large objects, ſeen by rays that fall upon reflecting or refracting ſurfaces, not only perpendicularly or almoſt perpendicularly, but with any degrees of obliquity.

PROPOSITION I.

598. WHEN *a large picture or a plane object of any given ſhape, ſtands perpendicularly to the common axis of any number of refracting or reflecting ſurfaces, its apparent ſhape, ſituation, magnitude and diſtance from the eye, at any point of the axis, may be found as follows.*

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* Art. 586.

Let *OABP* be the axis of the reflecting or refracting ſurfaces *AR* *BS*; *O* the place of the eye; *ORSQ* any reflected or refracted ray; *QP* a line perpendicular to the axis; *x* the interſection of a line *Qx*, drawn parallel to the axis, and of the viſual ray *OR* produced; Π*x* a curve line deſcribed by the interſection *x* while the viſual angle *AOR* is diminiſhed to nothing. Now while the whole figure is turned about the axis *OP*Π, the curve Π*x* will deſcribe a curve ſurface and *PQ* a circular plane. Let any part *abc d* of this plane be the given object or picture; and while an indefinite line is carried round the out-lines of the picture *abcd*, and is kept parallel to the axis *P*Π, it will trace the out-lines of a correſponding picture upon the curve ſurface at Π*x*. I ſay the plane picture *abcd* will appear by the reflected or refracted rays, in the ſame ſhape, ſituation, magnitude and diſtance, with which the curve picture at Π*x* would appear to the naked eye at *O*; ſuppoſing the eye to be inverted and turned about when the curve Π*x* falls behind it *, as in fig. 514. Fig. 510, 511. * Art. 130.

For imagine a phyſical circle deſcribed by the revolution of any point *Q*, about the axis *OP*, to be removed into the place of the equal circle deſcribed by the revolution of the correſponding point *x*; and if the ſurfaces *AR*, *BS* were taken away, its apparent ſhape, ſituation, magnitude and diſtance would be the ſame as before*. For ſince the viſual rays as *RO*, *xO* are always in one continued line, the ſhape, ſituation and magnitude of the picture upon the retina would be the ſame as before. And of conſequence all the phyſical circles that compoſe the plane at *PQ*, when thus removed till they touch the curve ſurface at Π*x*, would ſeverally and jointly appear to the naked eye at *O*, in the ſame ſhape, ſituation, magnitude and diſtance as they did before. What has been ſaid of the whole circles, is alſo true of any given parts of them that compoſe any part *abcd* of the circular plane at *PQ*; namely, that their ſeveral arches, which compoſe the plane figure *abcd*, when ſeverally removed, by a motion parallel to the axis *P*Π, till they touch the curve ſurface Π*x*, will there compoſe a correſponding part of it, which would appear to the naked eye at *O* in the ſame ſhape, ſituation magnitude and diſtance, with which the plane *abcd* appeared by the reflected or refracted rays. *Q. E. D.* * Art. 139.

599. *Corol.* 1. Hence if the plane ſurface at *PQ* and the curve ſurface at Π*x*, be both cut by any plane paſſing through the axis *P*Π or parallel to it, the ſtraight ſection of the former will appear in the ſame ſhape, ſituation, magnitude and diſtance, with which the curve ſection of the latter would appear to the naked eye remaining at *O*.

600. *Corol.* 2. Whenever the interſection *x* deſcribes a ſtraight line perpendicular to the axis *O*Π, the object at *PQ* will then appear in its proper ſhape; that is, in the ſame ſhape, ſituation, magnitude and di-

ſtance with which it would appear to the naked eye viewing it upon a perpendicular plane at the diſtance $O\Pi$.

LEMMA I.

601. *Let there be any number of indeterminate finite lines* x, y, z, *whoſe fluxions bear any finite ratio's to one another, and let another line* $v = \frac{xy}{z}$; *I ſay the fluxion of* v *will alſo bear a finite ratio to the fluxion of* x *or* y *or* z.

For by the known rule for finding fluxions we have $\dot{v} = \frac{\dot{x}y}{z} + \frac{\dot{y}x}{z} - \frac{\dot{z}xy}{zz}$ whoſe ſeveral parts bear finite ratio's to one another, as being compounded of none but finite ratio's; and conſequently the ratio of the whole value of $\dot{v}$ to any one part of it, is alſo finite; and therefore the ratio of $\dot{v}$ to $\dot{x}$ or $\dot{y}$ to $\dot{z}$ is alſo finite. *Q. E. D.*

LEMMA II.

Fig. 510, 511.

602. *If all the refracting or reflecting curves* AR, BS *cut their axis* $AB\Pi$ *at right angles, the curve* Πx *will alſo cut it at right angles.*

For let $R\alpha$, $S\beta$, $x\varpi$ be perpendiculars to the axis, and let the rays RS, SQ be produced till they cut it in φ and γ. Then ſince the triangles $O\alpha R$, $O\varpi x$ are ſimilar, and alſo $\alpha\varphi R$, $\beta\varphi S$, and likewiſe $\beta\gamma S$, $P\gamma Q$; we have $O\alpha : O\varpi :: \alpha R : \varpi x$ or PQ, that is, in a ratio compounded of αR to βS, βS to PQ, that is, of $\alpha\varphi$ to $\beta\varphi$, $\beta\gamma$ to $P\gamma$. Hence $O\varpi = O\alpha \times \frac{\beta\varphi}{\alpha\varphi} \times \frac{P\gamma}{\beta\gamma}$. Now when the angle AOR is diminiſhed to nothing, the interſections φ, γ will move to the focus's f, g and coincide with them; and $O\varpi$ will become $O\Pi = OA \times \frac{Bf}{Af} \times \frac{Pg}{Bg}$. Therefore when the abſciſs $\Pi\varpi$ is vaniſhing, the vaniſhing increments or decrements of the finite lines $O\alpha$, $\alpha\varphi$, $\beta\varphi$, $\beta\gamma$, $P\gamma$ are theſe reſpectively, αA, $\alpha A \pm \varphi f$, $\beta B \pm \varphi f$, $\beta B \pm \gamma g$, γg. But the aberration φf is as αA*, and γg as βB*, and conſequently all theſe vaniſhing increments or decrements bear finite ratio's to one another. Therefore $\varpi\Pi$ the vaniſhing increment or decrement of $O\varpi$, bears a finite ratio to αA, by the firſt Lemma, becauſe the vaniſhing increments of quantities are as their fluxions. But the ultimate ratio of αA to αR is a ratio of minority infinitely great, becauſe the curve AR, is ſuppoſed to cut its axis at right angles, and the ratio of αR to ϖx, being the ſame as $O\alpha$ to $O\varpi$, is finite. Conſequently the ultimate ratio of $\varpi\Pi$ to ϖx, being compounded of $\varpi\Pi$ to αA, αA to αR, αR to ϖx, is a ratio of minority infinitely great, and therefore the curve Πx cuts the axis of the other curves at right angles, which is alſo its own axis, becauſe the curves AR, BS are ſuppoſed to be ſimilar and equal on each ſide of it.

* Art. 645.

603.

603. *Corol.* 1. Hence it follows that the curve $\Pi\kappa$ cannot form an angle of contact with its axis at Π.

604. *Corol.* 2. The focus's f, g being given, the vertex Π of the curve $\Pi\kappa$ is found by taking $O\Pi = OA \times \frac{Bf}{Af} \times \frac{Pg}{Bg}$.

605. *Corol.* 3. Though the curves AB, BS were ſuch as could reflect or refract all the rays of a large pencil ſo accurately as to belong to the focus's f, g, yet by reaſon of the increments αA, βB the line $O\varpi$ would be variable, and conſequently the interſection κ would ſtill deſcribe a curve.

606. *Corol.* 4. Though the refracting curves were changed into their tangents, yet the interſection κ would ſtill deſcribe a curve, by reaſon of the aberrations $f\varphi$, $g\gamma$ by theſe refractions*. * Art. 647.

607. *Corol.* 5. But if the reflecting curves were changed into their tangents, the interſection κ would deſcribe a ſtraight line perpendicular to the axis at Π; there being no aberrations by theſe reflections.

608. *Corol.* 6. Therefore an object at PQ appears exactly in its proper ſhape by reflections from plane ſurfaces*. * Art. 600.

PROPOSITION II.

609. *The reflecting or refracting curves, and their intervals from each other and from the eye and object, being given; it is propoſed to find whether the object will appear convex or concave towards the eye.*

Things remaining as they were, let Ar, Bs be tangents to the curves AR, BS; and let one of the remoteſt rays OR cut the firſt tangent in r; join fr cutting the next tangent in s; join alſo gs cutting the object in L; and laſtly draw the reflected or refracted rays RS, SQ, either by an accurate conſtruction*, if needful, or only by obſerving which way their aberrations $f\varphi$, $g\gamma$ tend from the ſucceſſive focus's f, g; which will always be evident by the ſhapes of cauſticks deſcribed in Art. 69 &c. or 452 and 478 &c. Fig. 512 to 515. * Art. 469.

610. *Caſ.* 1. Then if it be evident from the poſition of the ray SQ and of the line sL, that PQ is longer than PL, and will continue ſo till the viſual angle AOR is reduced to nothing; the object PQ will appear convex towards the eye: and on the contrary, if PQ be always leſs than PL, the object will appear concave; and in both caſes more or leſs curved, *cæteris paribus*, as the ratio between PQ and PL is greater or leſs.

For let the lines $Q\kappa$, $L\lambda$, drawn parallel to the axis, cut the ray OR, (produced) in κ and λ; and let $\lambda\Pi$ be perpendicular to the axis; and in like manner as the value of $O\varpi$ was found by ſimilar triangles, in Lemma 2, we ſhall find $O\Pi = OA \times \frac{Bf}{Af} \times \frac{Pg}{Bg}$. Which being invariable, ſhews that while the angle $\Pi O\lambda$ is decreaſing to nothing, the interſection λ deſcribes a fixt perpendicular $\lambda\Pi$. The ſame point Π is alſo the vertex

of

* Art. 604. of the curve deſcribed by the other interſection $\varkappa$*, and conſequently $\Pi\lambda$ is a tangent to it by Lemma 2. Therefore if PQ be always longer than PL, and conſequently $O\varkappa$ always longer than $O\lambda$, the curve $\Pi\varkappa$ * Art. 603. muſt be convex towards the eye*. And on the contrary, if PQ be always ſhorter than PL, and conſequently $O\varkappa$ always ſhorter than $O\lambda$, the curve $\Pi\varkappa$ muſt be concave towards the eye.

611. *Caſ.* 2. Let SQ and sL (produced) interſect one another in d; and if the object be placed near their interſection, the curve deſcribed by d, while the angle AOR is diminiſhing, may croſs the object; and then the curve deſcribed by $\varkappa$ will alſo croſs its tangent $\Pi\lambda$ and form contrary flexures as repreſented ſeparately in Fig. 516. For when d touches the object the intervals LQ, $\lambda\varkappa$ will be reduced to nothing, and will become negative, lying on the other ſides of L and λ, after d has paſſed by the object. But by reaſon of the quick decreaſe of the angle LdQ or $gd\gamma$, the negative intervals LQ, $\lambda\varkappa$ will be become ſo ſmall that the contrary flexure of the curve $\Pi\varkappa$ may ſcarce be perceptible by the eye. And if ſo, the object muſt appear convex or concave towards the eye according as the greateſt PQ cut off by the remoteſt ray SQ, is longer or ſhorter than the greateſt PL, as in the foregoing caſe.

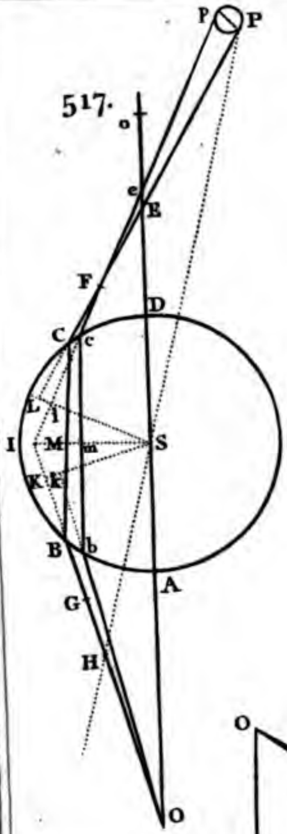
612. *Caſ.* 3. Conſequently if the object touches the interſection d when remoteſt from the axis, it may be reckoned to appear ſtraight in that ſituation or ſomewhere very near it: which is farther evident, becauſe the object being carried for a conſiderable ſpace from one ſide of d to the other, muſt change its apparent figure from concave to convex, or on the contrary by caſe 1.

Fig. 514. 613. *Caſ.* 4. If the object is ſo placed that the points γ, g are on contrary ſides of the point P, the points Q, L muſt alſo be on contrary ſides of it. But by diminiſhing the angle AOR, the point γ in approaching towards g will paſs by the point P, and then the points Q, L will be both on the ſame ſide of the axis; and ſo the rule in caſe 1 will ſoon take place. But the object will appear in a manner double, as might eaſily be explained, and ſo confuſed that its apparent ſhape will ſcarce be diſcernible even through a pin-hole.

Therefore the rule in the firſt of theſe caſes is generally ſufficient for determining the apparent convexity or concavity of an object, eſpecially when it appears moſt evidently ſo. Q. E. D.

Fig 512 to 515. 614. *Corol.* Hence if we imagine a fictitious kind of rays as $OrsL$ to flow from the eye, and to be all reflected or refracted at the tangents Ar, Bs ſo accurately as to belong to the ſucceſſive focus's f, g; the object PQ ſeen by ſuch rays returning back in the ſame lines $LsrO$, would * Art. 600. always appear accurately in its proper ſhape at the tangent $\Pi\lambda$\*. The deformed appearances of objects are therefore cauſed both by the deflections of the curves AR, BS from their tangents and by the aberrations

517.



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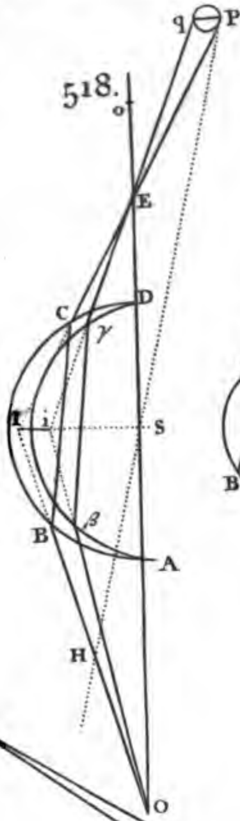

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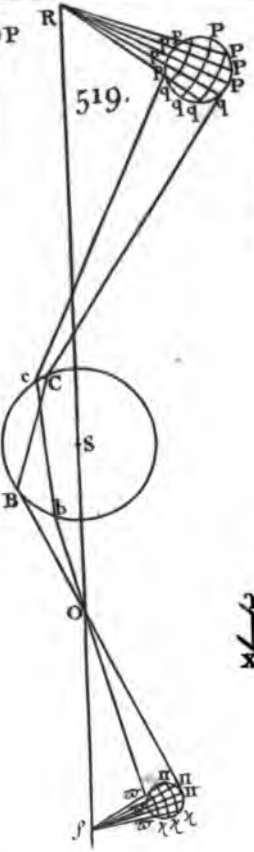

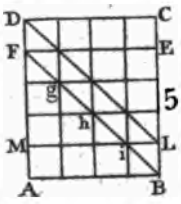

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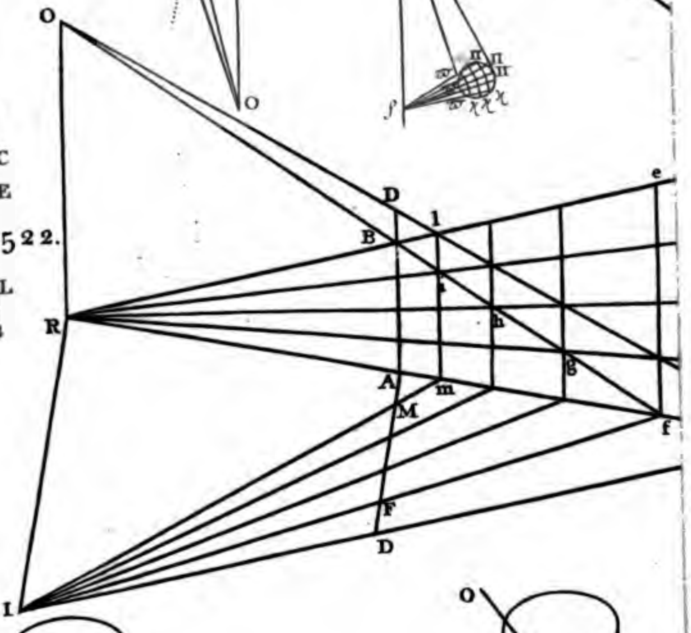

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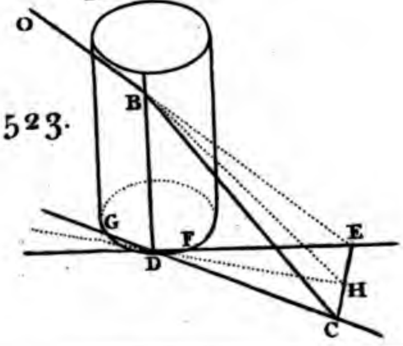
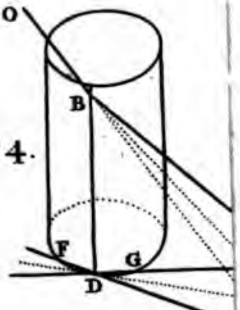


524.



tions of the rays. Upon both theſe accounts the minuter parts of large objects lying in order from the axis, become unequally or rather improperly magnified or diminiſhed, and therefore appear at improper diſtances from the eye* and from each other; and thus the plane of the object is changed into a curve ſurface. For while the viſual angle AOx or AOR decreaſes uniformly, the equal lines PL, $\Pi\lambda$ decreaſe uniformly very nearly, or rather properly as they ſhould do in viſion with the naked eye, but PQ does not; therefore improper decrements of PQ appear under equal angles, and conſequently are improperly magnified or diminiſhed. * Art. 138. 139.

PROPOSITION III.

615. *Let the plane of this ſcheme repreſent a plane paſſing through the eye at* O, *and the center* S *of a refracting ſphere* ABCD; *and let it be produced till it cuts the ſun's disk in any line* Pp; *and out of an infinite number of rays ſuppoſed to flow in this plane from the eye at* O, *upon the great circle* ABCD, *let two of them, as* OBCFP, ObcFp, *fall upon the extremities of the chord* Pp *after refractions at the arches* Bb, Cc, *and after having croſſed each other in the focus* F; *then from the center* S *let* SkK *be perpendiculars to the incident rays* Ob, OB *produced, and* SlL *perpendiculars to the emergent rays; and while the quantity of the refractions at* Bb, Cc, *is varied by any lateral motion of the ſun, eye or ſphere, in that plane of refraction, the apparent magnitude of the given chord* Pp *will vary directly as* FL *and inverſely as* OK *very nearly, if the diameter of the ſphere be very ſmall in compariſon to its diſtances from the eye and ſun.* Fig. 517.

For then the angle LFl or PFp being but little more than half a degree at moſt, the angle KOk will alſo be but ſmall, and conſequently their perpendicular ſubtenſes Kk, Ll will be very nearly equal. For drawing SMm perpendiculars to the rays BC, bc within the ſphere; we have Mm to Kk and alſo to Ll, in the given ratio of the ſines SM, SK or SL that determine the refractions*. Therefore Kk equals Ll and the angle KOk is to the angle LFl as FL to OK*. But ſince the chord Pp is given, the angle PFp or LFl is very nearly invariable, by reaſon of the immenſe diſtance and ſpherical figure of the ſun; and conſequently the apparent magnitude of Pp, meaſured by the angle KOk, is as FL directly and OK inverſely very nearly. *Q. E. D.* * Art. 414. * Art. 222.

PROPOSITION IV.

616. *Things remaining as they were, let the ray* CP *cut the axis* OS *in* E, *and be produced till it meets the ray* OB *produced in* I; *and let the conical ſurface deſcribed by the ray* CEP, *turned about the axis* OS *into the poſition* γEq, *cut the disk of the ſun in the arch* Pq; *and while the refractions of the viſual rays* PCBO, qγβO, *that lye in this conical ſurface, are varied by any motion of the ſun, the eye or the ſphere, the apparent magnitude of the given arch* Pq *will vary directly as* IE *and inverſely as* IO *very nearly.* Fig. 518.

For let the point *I* deſcribe the arch *Ii* by the revolution aforeſaid; and ſince it is a common ſubtenſe of the angles *IOi*, *IEi*; the angle *IOi* is to *IEi* or *PEq* as *IE* to *IO*. But when the arch *Pq* is given, the angle *PEq* is very nearly invariable, by reaſon of the great diſtances of the ſun and eye from the ſphere, and conſequently the apparent magnitude of *Pq*, meaſured by the angle *IOi*, is as *IE* directly and *IO* inverſely very nearly. *Q.E.D.*

PROPOSITION V.

Fig. 517, 518. 617. *Things remaining as they were, let the refractions of the viſual rays be varied by any motion of the ſun, the eye or the ſphere; and the apparent magnitude of the ſun's disk will vary directly as the rectangle under* FL, EI, *and inverſely as the rectangle under* OK, OI *very nearly.*

Fig. 519. For let innumerable planes of incidence and refraction cut the ſun's diſk in the lines *Pp*, *Pp*, &c. all converging to a point *R* in the axis *OS* produced; and let the ſame planes cut the picture of his diſk upon the retina in as many correſponding lines Πϖ, Πϖ, &c. all converging to the point ρ in the ſaid axis produced backwards. Again let innumerable conical ſurfaces, conceived to be deſcribed by the revolution of innumerable viſual rays about the axis *OS*, cut the diſk in the arches *Pq*, *Pq*, &c. and its picture upon the retina in as many correſponding arches Πϰ, Πϰ, &c. Then ſince every one of the lines Πϖ, in the whole picture, is as the apparent magnitude of the correſponding lines *Pp**, in the whole diſk, that is as $\frac{FL}{OK}$*; the magnitude of the picture would alſo be as $\frac{FL}{OK}$, if the arches Πϰ were invariable. And in like manner ſince every one of the arches Πϰ in the ſame picture, is as the apparent magnitude of the correſponding arches *Pq* in the diſk, that is as $\frac{IE}{IO}$*; the magnitude of the picture would alſo be as $\frac{IE}{IO}$, if the lines Πϖ were invariable. Therefore when the lines Πϖ and the arches Πϰ both vary, it is eaſy to underſtand that the magnitude of the ſun's picture upon the retina, is as $\frac{FL}{OK} \times \frac{IE}{IO}$, that is, as the rectangle under *FL*, *IE* directly, and as the rectangle under *OK*, *IO* inverſely very nearly. *Q. E. D.*

* Art. 91.
* Art. 615.
* Art. 616.

Fig. 517, 518. 618. *Corol.* 1. Let the refracting ſphere be moved ſideways from the rays that come directly from the ſun to the eye, ſo as to deſcribe a circle about the eye in any one plane of incidence and refraction; and if the diameter of the ſphere be but ſmall in compariſon to its diſtances from the eye and the ſun, the apparent magnitude of the given chord or diameter *Pp* will decreaſe perpetually, and very nearly as *FL* does, till it vaniſhes when the viſual rays are tangents to the ſphere. For when the viſual ray *OB* touches the ſphere, the line *OK* becomes equal to *OB*, and is then the leaſt of all, and yet will differ but very little from its greateſt magnitude *OS*. Therefore *OK* may be reckoned invariable, and conſequently the apparent magnitude of Pp is as FL*; which decreaſes perpetually and very faſt till it be reduced to nothing[a].

619. *Corol.* 2. Upon the ſame ſuppoſitions, the apparent magnitude of any given arch Pq upon the ſun's diſk, and conſequently of the chord or diameter which joins its extremities, will decreaſe perpetually, and very nearly as IE does[b]. For when the angle which the refracting ſphere ſubtends at the eye is but ſmall, IE decreaſes perpetually during the motion above mentioned[c].

620. *Corol.* 3. Hence upon the ſame ſuppoſitions as before, the apparent magnitude of the ſun's diſk will decreaſe perpetually and very nearly as the rectangle under FL, IE. And its apparent ſhape will grow more and more oval while the ſphere is moved ſideways more and more; its diameter Pp, in the plane of incidence and refraction, appearing ſhorter than the diameter perpendicular to it. Becauſe FL decreaſes faſter than IE; being always contained within it, except when they are both ſituated in the axis OS, and then they are equal to one another.

621. *Corol.* 4. Let o be the conjugate focus to O, and the area of the oval picture of the ſun upon the retina, will be to its circular area when the ſphere is exactly between the eye and the ſun, as $FL \times IE$ to So^2. For when the ſphere is exactly between the eye and ſun, FL and IE become equal to each other and to So.

622. *Corol.* 5. In the ray $PCBGO$ let G be the focus of the rays that flow from P in any one plane of incidence and emergence; and H the focus of the rays that flow from P in the conical ſurfaces deſcribed by the ſame ray $PCBHO$ turned about the line PSH, the point H being the vertex of the cones; and the denſity of all the rays flowing from P, where they fall upon the cornea of the eye at O, will be as $GK \times HI$ directly, and as $GO \times OH$ inverſely[d]; and conſequently as $GK \times HI$, when the ſphere is ſo remote as to ſubtend but a ſmall angle at the eye.

623. *Corol.* 6. Hence when a remote ſphere ſubtends but a ſmall angle at the eye, the ſun's apparent brightneſs, ſeen through any part of it, is invariable. For the apparent brightneſs of any object is as the denſity of the rays in any one pencil directly, and as its apparent magnitude inverſely; that is as $\frac{GK \times HI}{FL \times EI}$*. But ſince the rays that come from O and P, fall parallel or nearly parallel upon the circle $ABCD$, the lines GK and FL are nearly equal; and ſo are HI and IE. For the line SI biſects the angle EIH, and ſince SE is ſuppoſed to be nearly parallel to IH, and SH to IE, the angle EIS equals ISH, and HIS equals ISE; and conſequently the triangles EIS, HIS are equicrural and equal to one another very nearly.

624. *Corol.* 7. Hence when the centers of innumerable refracting ſpheres are placed at equal intervals from one another in a large ſpherical ſurface whoſe center is the eye at O; this ſurface will appear illuminated by in-

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* Art. 615.

[a] Art. 439.

[b] Art. 616.

[c] Art. 514.

[d] Art. 521.

* Art. 622. 620.

numerable images of the ſun, all of the ſame brightneſs; but leſs and leſs in magnitude as they lye remoter from the ſun. And conſequently if the magnitude of the ſpheres be diminiſhed and their number be increaſed to infinity, the light of that ſpherical ſurface will appear uninterrupted, and ſtrongeſt next the ſun, and weaker perpetually in the parts that lye remoter from him; which agrees with our former demonſtration in the 525th article.

625. *Corol.* 8. What has been demonſtrated of the ſun is applicable to the moon, or to any round body, ſo ſmall or ſo remote from the ſphere as to ſubtend but a ſmall angle at any point of it; and is alſo agreeable to experience in viewing a lighted candle through a round bellyed jugg or decanter full of water, while it is moved ſideways from the rays that come directly to the eye.

626. *Corol.* 9. While the apparent magnitude of the candle decreaſes, its apparent diſtance will be found to increaſe, and on the contrary; and ſo its ſucceſſive apparent places will appear to deſcribe a curve; which in the caſe abovementioned[a] is convex towards the eye. And the apparent diſtance of the candle will vary reciprocally in a ſubduplicate ratio of the apparent magnitude of its ſurface, that is, reciprocally as a middle proportional between *FL* and *IE**; or rather reciprocally as *IE*, that is reciprocally as the apparent height of the candle[b]; becauſe its apparent breadth appears oblique to the viſual rays, by reaſon of the curvity of its ſucceſſive apparent places.

[a] Art. 625.
* Art. 620.
[b] Art. 616. 602.

627. *Corol.* 10. For this reaſon any large plane ſurface, like a ſaſh-window, viewed through a ſphere will not appear plane but convex towards the eye. And beſides this convexity towards the eye, the top and bottom and ſides, and alſo the wooden bars that contain the glaſs, will appear concave towards the middle of the window, where a line drawn from the eye through the center of the ſphere would fall upon it. So that the whole appearance will be like the meridians upon a globe to the naked eye. For ſince the interior ſquares of glaſs are more magnified than the exterior ones, the intervals between the bars will be gradually diminiſhed in going outwards from the middle; which agrees with the manner of deſcribing theſe appearances by the laſt propoſition.

PROPOSITION VI.

Fig. 520. 628. *Any line* Pp *either ſtraight or crooked, which is ſeen from* O *by rays reflected from a ſtraight line* Bb, *drawn upon a plane ſurface* ABCcbA, *or upon the ſide of a cone or a cylinder, will appear in its proper ſhape; and the apparent diſtance of any point of it from the eye, will be equal to the whole courſe of the viſual ray.*

For let the reflecting plane *ABCcbA* be generated by the motion of the line *ABC* about the axis *OAo* perpendicular to it. And let a ſuperficial pencil of rays OBb be ſuppoſed to flow from the eye at O, and after reflection from the ſtraight line Bb let them fall upon the ſurface of an object in the line Pp, either ſtraight or crooked. From every point of the object Pp draw the lines $P\Pi, p\varpi$ parallel to OA till they meet the viſual rays produced in Π, ϖ; and the apparent object $\Pi\varpi$ compoſed of all their interſections will be ſimilar and equal to Pp. For taking Ao equal to AO, all the reflected rays will flow from o[*], and ſince all the triangles OBo, Obo, are equicrural, all the triangles $PB\Pi$, $Pb\varpi$, being ſimilar to the former, will alſo be equicrural. Therefore the compound line $OB+BP$ equals $OB\Pi$ equals oBP, and likewiſe $Ob\varpi$ equals obp; and conſequently the plane figure $O\Pi\varpi$, conſiſting of all the former lines, will be ſimilar and equal to the figure oPp conſiſting of all the latter. Therefore the Line $\Pi\varpi$ is ſimilar and equal to Pp; and conſequently is ſeen by reflection in the ſame place and ſhape as $\Pi\varpi$ would appear in, to the naked eye[a].

Fig. 520. The ſurface of a cone is generated by the revolution of one ſide of an angle about its other ſide, and the ſurface of a cylinder by the revolution of one ſide of a parallelogram about its oppoſite ſide. Conſequently ſuppoſing the reflecting line Bb to be laid upon either of theſe revolving lines, ſo as to coincide with it, the reflecting plane BAb may be conceived to touch the conical or cylindrical ſurface in the line Bb; and the reflections from Bb being the ſame whether they be made from the curve ſurfaces or from the tangent plane, it follows that the line Pp will appear ſtraight, or in its proper ſhape, by reflection from the cone or cylinder, as well as from the tangent plane. *Q. E. D.*

629. *Corol.* 1. Hence one may determine the apparent magnitudes, diſtances and ſhapes of a ſeries of given objects ſeen by reflection from a cylinder, in this manner. Let the circle $ABCD$, whoſe center is E, be the baſe of an upright cylinder, or rather a circular ſection of it, parallel to its baſe and paſſing through the eye at O; and let this plane be extended and cut a row of trees, for example, in P, R, T, X, &c. Join OE and let $o\alpha\beta\gamma\delta$ be a cauſtick by reflection of innumerable rays ſuppoſed to flow from O; and let the lines $P\alpha$, $R\beta$, $T\gamma$, $X\delta$, &c. be ſo drawn as to touch the cauſtick in α, β, γ, δ, &c. and let them cut the reflecting circle in A, B, C, D, &c, join OA, OB, OC, OD, &c. and in each of them produced, take $A\varpi=AP$, $B\rho=BR$, $C\tau=CT$, $D\varphi=DX$, &c. Then ſuppoſing the trees PQ, RS, TV, XY, &c. to be removed and planted upright at $\varpi\kappa$, $\rho\sigma$, $\tau\upsilon$, $\varphi\chi$, &c. they would appear to the naked eye at O, if the cylinder was removed, of the ſame magnitudes, at the ſame diſtances and in the ſame ſhapes, as they appeared in by reflection from the cylinder when they ſtood in their proper places. And if regular curves be drawn through the points ϖ, ρ, τ, φ, &c. and through κ, σ, υ, χ, the crooked ſurface that lyes between theſe curves, will be the apparent place and ſhape of the ſurface $PQXY$. But the readieſt practical way is firſt to draw any ray OB and then its reflection BR, cutting PX in any point R, and to take $B\varrho=BR$ &c.

Fig. 521.

[*] Art. 202.

[a] Art. 139.

630. *Corol.* 2. In like manner if another row of trees be parallel to the former, either on the ſame ſide of the cylinder or on the oppoſite one; the ſpace that lyes between the repreſentations of the baſes of both rows, will be the apparent figure of the walk between the rows. Thus we have determined the repreſentations of any planes either perpendicular or parallel to the baſe of the cylinder. Conſequently the repreſentation of any oblique plane, as of one ſide of a roof of a houſe or the like, may be found by determining the repreſentations of the top and bottom of it.

The manner of drawing deformed pictures, which ſhall appear regular in a cylinder placed upon the plane of the draught, is partly the reverſe of the foregoing method. In order to deſcribe it more diſtinctly I premiſe the following lemma's.

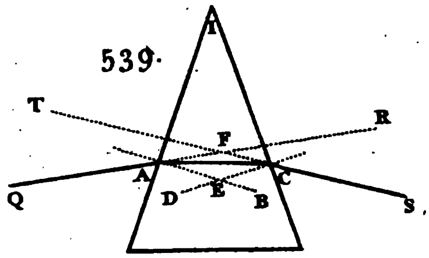
LEMMA I.

Fig. 522. 631. A rectangular parallelogram *ABCD* being divided into any number of little parallelograms, and being placed upright upon a plane *ABR*, (the line *AB* in one figure being placed upon *AB* in the other), it is propoſed to deſcribe its ſhadow *ABcd*, caſt upon that plane by rays flowing from a given point *O*, placed at a given height *RO* above the plane of the ſhadow.

From the point *R* which lyes directly under the luminous point *O*, through the extremities and the diviſions of the baſe *AB*, draw the lines *RAd*, *RBc*, &c. Then placing the line *AD*, equal to the height of the given parallelogram, and alſo *RI* equal to the height of the luminous point, upon the plane of the ſhadow, at right angles to *RA*; from *I* through the extremities and diviſions of *AD*, draw *IDd*, *IFf*, *IMm*, &c. cutting the line *RA* produced in *d*, *f*, *m*, &c. And through theſe points *d*, *f*, *m*, &c. draw the lines *dc*, *fe*, *ml*, &c. parallel to *AB*; and the trapezium *ABcd* ſo divided into leſſer trapeziums, will be the ſhadow of the parallelogram *ABCD* and of its diviſions.

For conceiving the plane of the triangle *IRd* with all its lines, to be turned about the line *RAd*, till it ſtands at right angles to the plane *RAB*, the point *I* will then coincide with the luminous point in the air, and the diviſions of the line *Ad* will be the ſhadows of the diviſions of the ſide *AD* of the given parallelogram. And for the like reaſon the lines extended from *R* are the ſhadows of all the lines in the parallelogram which are perpendicular to *AB*; and the ſhadow *fe* of any line *FE*, parallel to *AB*, being the common interſection of two planes paſſing through the parallels *AB*, *FE*, is alſo parallel to *AB*. *Q. E. D.*

632. *Corol.* 1. If the given rectangle *ABCD* be contracted to a ſquare *ABEF*, divided into any number of leſſer ſquares, its ſhadow may be found more expeditiouſly; by drawing *RO* parallel to *AB* and equal to the height of *O*, and then by drawing *OBf*, cutting the ſhadows of the perpen-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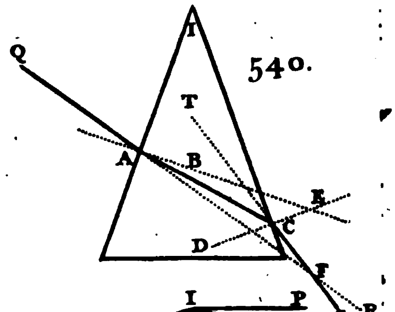
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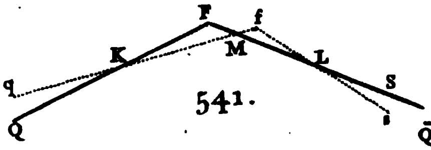
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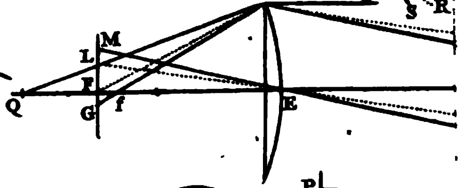

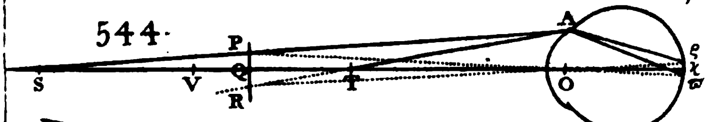
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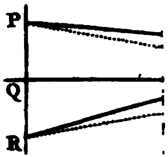

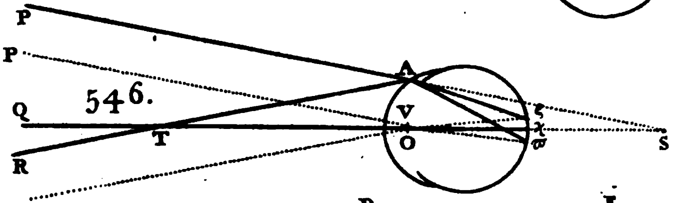
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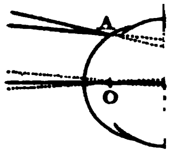

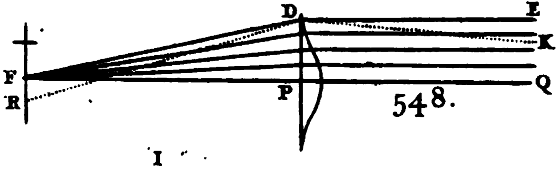
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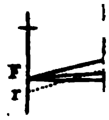

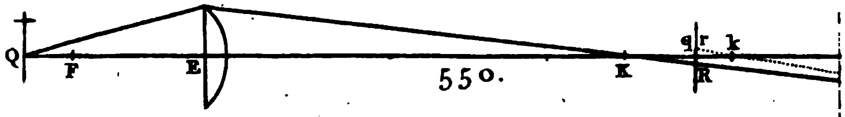
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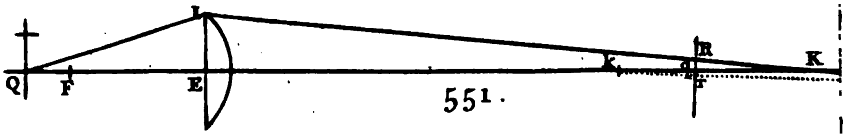
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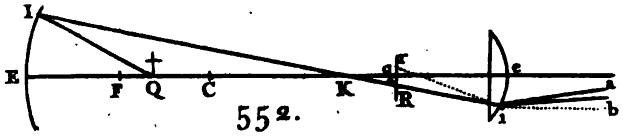
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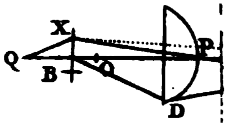

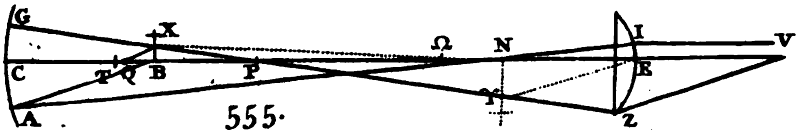
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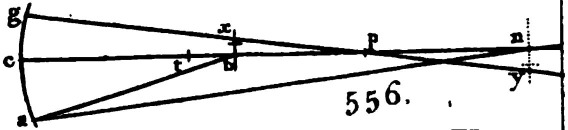
556.

perpendicular lines in the points *f*, *g*, *h*, *i*, through which the parallel ſhadows muſt paſs. For ſince *AF* equals *AB*, the point *f* is the ſame in both conſtructions; becauſe *Rf* is to *Af* as *RI* to *AF* or as *RO* to *AB*. And ſince *BF* is a diagonal of all the little ſquares, its ſhadow *Bf* muſt be a diagonal of all their ſhadows.

633. *Corol.* 2. This conſtruction is alſo applicable to a parallelogram, provided it be divided into little ſquares; by placing its ſide *AD* along the baſe *AB*, produced if need be, and by drawing *OD* cutting *RA* and *RB* produced in *d* and *l*. For *dl* is the ſhadow of the diagonal *DL* of the ſquare *DCLM*, cut off from the parallelogram *ABCD*.

634. *Corol.* 3. Hence it follows that the ſhadow of the center of any ſquare, may be found by drawing the diagonals of the correſponding trapezium.

635. *Corol.* 4. If the length of the ſhadow *dA* and the point *R* be given or aſſumed, the luminous point *O* may be found by drawing *dD* which will cut off the perpendicular *RO* required.

636. *Corol.* 5. Hence we may underſtand the method of drawing a copy of a picture, which ſhall appear deformed when viewed directly, and yet ſhall appear in juſt proportion when viewed from a certain point. For having deſcribed a parallelogram or a ſquare about the original picture, and having divided it into any number of leſſer ſquares, the more the better, and laſtly having divided another ſquare *ABEF* of any ſize into the ſame number of leſſer ſquares; firſt project its ſhadow *ABef*, and then transfer the parts and lineaments and colouring in every little ſquare of the original picture, into the correſponding trapeziums of the ſhadow *ABef*. For to the eye of a ſpectator placed in the luminous point *O*, the deformed picture will appear reformed into the true proportions of the original; becauſe the ſhape of the image of the whole picture and of all its parts upon the retina, is the ſame as if the rays had come to the eye originally from a regular picture drawn in miniature upon a ſquare placed upright upon its baſe *AB*. And if the ſhades and colourings in the deformation be the ſame alſo as in the original picture, the ſpectator will judge it to be regular rather than deformed; by being more accuſtomed to view regular pictures in an upright poſition than deformed ones in an oblique poſition.

LEMMA II.

637. When a ray *OB* falls obliquely at *B* upon the convex or concave ſurface of an upright cylinder *BFG*, it is propoſed to find the point *C* where the reflected ray *BC* will cut the plane of the baſe of the cylinder. Fig. 523, 524.

Draw *BD* perpendicular to the baſe cutting its circumference *FG* in *D*, and let *DH* be a tangent to it at *D*; then let the incident ray *OB* be produced till it cuts the plane of the baſe produced in *E*, and draw *EH* perpendicular to *DH*, and produce it till *CH* equals *HE*, and *C* will be the point where the reflected rays *BC* will fall upon the plane of the baſe produced.

For joining *DC*, *DE* and alſo *BH*, the plane of the triangle *DBH* will touch the cylindrical ſurface in the line *BD*; and ſince the perpendiculars *HC*, *HE* to this plane are equal, if the point *C* be ſuppoſed a focus of incident rays upon this plane produced, the point *E* will be their focus after reflection*. Therefore on the contrary, the ray *OB* tending towards *E*, will be reflected to *C*; and the reflection from the tangent plane is the ſame as from the cylindrick ſurface at *B**. *Q. E. J.*

* Art. 202.

* Art. 19.

638. *Corol.* 1. A more commodious practice in order to the ſolution of the following problem, is to produce *OB* to *E*, and to draw *DE* cutting the circumference of the baſe in *F*, and to inſcribe in it a chord *DG* equal to *DF*; and in *DG* produced to take *DC* equal to *DE*, which gives the point *C* where the reflected ray will fall upon the plane of the baſe. Becauſe the equal chords *DF*, *DG* are equally inclined to the tangent *DH*.

Fig. 525.

639. *Corol.* 2. Hence ſeveral rays flowing from *O* upon the line *BD*, drawn along the ſide of the cylinder, will all be reflected to the ſtraight line *DC*; where their intervals are equal to the reſpective intervals of the incident rays produced till they cut the line *DE*. Becauſe each two diſtances *DC*, *DE* are equal.

640. *Corol.* 3. Several rays flowing from *C* upon *BD*, will all diverge from *E*; and therefore but one of them can fall upon the ſame point *O*.

641. *Corol.* 4. Any incident and reflected ray *OB*, *BC* produced, make equal angles with the line *BD* produced, and alſo with the plane of the baſe. For the triangles *BDC*, *BDE* are equal.

PROPOSITION VII.

Fig. 526 to 529.

642. *To paint upon a plane a deformed copy* ABCDEKIHGF, *of an original picture, which ſhall appear regular when ſeen from a given point* O, *by rays reflected from a poliſhed cylinder, placed upon the circle* lnp, *equal to its given baſe.*

From the point *R* which lyes directly under *O* the place of the eye, draw two lines *Ra*, *Re*, which ſhall either touch the baſe of the cylinder, or elſe cut off two ſmall equal ſegments from the ſides of it; according as the copy is intended to be more or leſs deformed. Then taking the eye raiſed above *R* to the given height *RO*, ſomewhat greater than that of the cylinder, for a luminous point; deſcribe the ſhadow *aekf* of a ſquare *aexz* (or parallelogram), ſtanding upright upon its baſe *ae* any where behind the arch *lnp*, and divided after a ſimilar manner to another ſquare (or ſimilar parallelogram) drawn upon the original picture[a]. Let the lines drawn from *R* to the extremities and diviſions of the baſe *a*, *b*, *c*, *d*, *e*, cut the remoteſt parallel ſhadow in the points *f*, *g*, *h*, *i*, *k*, and the arch of the baſe in *l*, *m*, *n*, *o*, *p*;

[a] Art. 631.

n, o, p; from which points draw the lines lAF, mBG, nCH, oDI, pEK, as if they were rays of light that came from a focus R, and were reflected from the baſe lnp; ſo that each couple as lA, lR produced may cut off equal ſegments from the circle. Laſtly transfer the lines laf, mbg, &c. and all their parts in the ſame order, upon the reſpective lines lAF, mBG, &c. and having drawn regular curves, by eſtimation, through the points A, B, C, D, E, and through F, G, H, I, K, and through every intermediate order of points; the figure $ACEKHF$ ſo divided will be the deformed copy of the ſquare drawn and divided upon the original picture; and will appear ſimilar to it when ſeen in the poliſhed cylinder, placed upon the baſe lnp, by the eye put into its given place O. And the ſame directions that were given in art. 637, ſerve for painting the deformed copy of the picture upon the deformed ſquare AK.

For ſuppoſing the eye at O, raiſed to the height RO, to be a luminous point; every ſuperficies of rays emitted from it towards every line of the ſquare and conſequently towards every line of its ſhadow ak, being intercepted by the ſurface of the cylinder, will be reflected to every correſponding line of the deformation AK; as will eaſily appear by comparing the ſolution of the lemmas and their corollaries with that of this problem. Therefore on the contrary, the rays which flow backwards from every line of the deformation AK, will be reflected to the eye as if they came directly from the lines of the ſhadow ak or from the lines of the ſquare it ſelf. *Q. E. D.*

643. *Corol.* 1. If the ratio of the ſides of the original picture, and conſequently of the ſimilar parallelogram $aex\zeta$, be given; and nR be given or aſſumed; and nt, the height of the cylinder, or of the higheſt point of reflection, be alſo given; the height of the eye may be determined by placing nt perpendicular to nR, and by placing the parallelogram $aex\zeta$ upon its baſe ace at any convenient diſtance behind the arch lp, and by producing its baſe ca till $c\theta$ be equal to its height; then the line θt produced will cut off the required height RO; as will appear by conceiving the triangle $ROt\theta b$ to ſtand perpendicular to the paper upon the line $Rncb$. And if the data be varied, the reſt may be eaſily determined by the relation of the lines we have been conſidering. Fig. 530.

644. *Corol.* 2. If the portion of the cylindrical ſurface, which reflects the light, be extended into a plane, the deformation AK will become exactly equal to the ſhadow ak. The 527, 528, and 529th figures belong to a concave cylindrical ſurface.

CHAPTER XIII.

The theory of the aberrations of rays is reſumed and carried farther, in order to diſcover the limits of perfection of reflecting and refracting Microſcopes; and to determine for them what was determined for Teleſcopes in the ſeventh chapter.

PROPOSITION I.

645. HAVING *the focus of homogeneal rays incident upon a ſpherical ſurface, to find the aberrations of the refracted or the reflected rays.*

Fig. 531, 532. Let Q and q be conjugate focuſes of incident and refracted or reflected rays[a]; QI an incident ray upon the ſpherical ſurface IO, whoſe ſemidiameter is SO; IK the refracted or reflected ray (produced) cutting the axis $OSqQ$ in K; IX the ſine of the arch IO; and for the diſtances OQ, OS, OX putting Q, S, X; and m to n for the ratio of refraction, m being bigger than n; and θ for $m-n$; I ſay, if the ray be refracted at I, the aberration $qK = \overline{\frac{Q-S}{Q+\frac{n}{\theta}S}}\Big|^2 \times \frac{nnQ-nnS-mnS}{\theta mQ}X$; and if reflected, $qK = \overline{\frac{Q-S}{Q+\frac{1}{2}S}}\Big|^2 \times \frac{1}{2}X$.

[a] Art. 211.

Fig. 533. Draw $BSZM$ parallel to IQ, cutting IK produced in Z; and let SA be perpendicular to QI; and let t and M be the focuſes of refracted rays, ſuppoſing the incident ones to have come parallel and the neareſt to the ſemidiameters SO, SB; and by art. 224, $SM = St = \frac{n}{\theta}S$; and if the ray QI was parallel to SO, then would its aberration $ty = \frac{nn}{m\theta}X$ by art. 329; and by art. 333 the aberration $MZ : ty :: SA$ ſquare $: IX$ ſquare $:: QS$ ſquare $: QI$ ſquare or QO ſquare*. Hence MZ or $Z = \frac{\overline{Q-S}|^2}{QQ} \times \frac{nn}{m\theta}X$; and $SZ = \frac{n}{\theta}S - Z$. Now becauſe the triangles QKI, SKZ are ſimilar; it is as $QK : SK :: QI : SZ$; and conjointly as $QK : QS :: QI : QI + SZ$. Whence putting I for QI, we have $QK = \frac{\overline{Q-S} \times I}{I + \frac{n}{\theta}S - Z}$, and therefore when the point I comes to O and K to q, the conjugate focus to Q, we have $Qq = \frac{\overline{Q-S} \times Q}{Q + \frac{n}{\theta}S}$; and ſo the aberration $qK = (QK - Qq =) \overline{Q-S} \times \frac{I}{I + \frac{n}{\theta}S - Z} -$

* Art. 204.

Q

$\frac{Q}{Q+\frac{n}{\theta}S} = \overline{Q-S}\times\frac{QZ-\frac{n}{\theta}S\times\overline{Q-I}}{\overline{I+\frac{n}{\theta}S-Z}\times\overline{Q+\frac{n}{\theta}S}}$. With the ſemidiameter QI deſcribe the arch IE cutting SO in E; and calling OE, E, we have $Q-I=E$ and $I=Q-E$; ſo then $qK = \frac{\overline{Q-S}\times\overline{QZ-\frac{n}{\theta}SE}}{\overline{Q+\frac{n}{\theta}S-Z-E}\times\overline{Q+\frac{n}{\theta}S}} = \frac{Q-S}{\overline{Q+\frac{n}{\theta}S}\big|^2}\times\overline{QZ-\frac{n}{\theta}SE}$, becauſe $\overline{Q+\frac{n}{\theta}S}\big|^2$ is vaſtly greater than $\overline{Q+\frac{n}{\theta}S}\times\overline{Z+E}$. Now by art. 328, $XO:XE::QE:SO$, and disjointly $XO:EO::QE$ or $QO:QS$. Whence $E=\frac{Q-S}{Q}\times X$. For E and Z ſubſtitute their values, and the aberration $qK = \overline{\frac{Q-S}{Q+\frac{n}{\theta}S}}\Big|^2\times\frac{nnQ-nnS-mnS}{\theta mQ}X$.

Now if the ray be reflected at I, for n ſubſtitute $-m$ and conſequently $2m$ for θ, and the theorem changes into this, $qK = \overline{\frac{Q-S}{Q-\frac{1}{2}S}}\Big|^2\times\frac{1}{2}X$. For the calculation is the ſame whether the refracted ray goes backward or forward in the line IK; and to change the angle of refraction SIK into an angle of reflection, it (and its ſine n) muſt be diminiſhed to nothing, and then be made negative and equal to ($-m$ the ſine of) the angle of incidence SIQ. And during this change the method of calculation continues unaltered. Q. E. J. Fig. 532.

646. *Corol.* 1. Put Y for IX, and for a refracted ray the aberration $qK = \overline{\frac{Q-S}{Q+\frac{n}{\theta}S}}\Big|^2\times\frac{nnQ-nnS-mnS}{2\theta mQS}YY$, and for a reflected ray $qK = \overline{\frac{Q-S}{Q-\frac{1}{2}S}}\Big|^2\times\frac{1}{4}\frac{YY}{S}$. For $OX:XY::XY:2OS$ very nearly.

647. *Corol.* 2. Let S become infinite, and by the laſt corollary the aberration of a ray refracted at a plane ſurface, that is $qK=\frac{mm-nn}{mn}\times\frac{YY}{2Q}$; and the aberration of a ray reflected at a plane ſurface is nothing at all.

648. *Corol.* 3. When the point of incidence is given, the longitudinal aberration qK of a reflected ray, from its focus q, is as Sq^2, the ſquare of the diſtance of that focus from the center of the ſurface. For by this propoſition qK is to $\frac{1}{2}X$, the aberration from T the focus of parallel rays[a], as QS^2 to QT^2, or as Sq^2 to ST^2; becauſe QT, ST, qT, being continual proportionals[b], are proportionable to their ſums or to their differences.

[a] Art. 334.

[b] Art. 207.

649.

649. *Corol.* 4. When *Q* and *S* are given the longitudinal aberration of the outmoſt reflected or refracted ray is as YY, the ſquare of the ſemiaperture of the ſurface, by corol. 1.

LEMMA I.

650. When the terms of a ratio involve two ſorts of quantities, one of which is incomparably ſmaller than the other; the magnitude of that ratio is not altered by any alterations made in thoſe infinitely ſmall quantities.

This will be evident when applyed.

LEMMA II.

Fig. 534. 651. In the produced ſemidiameter or of the ſpherical ſurface oE, let ϖ and p be conjugate focuſes of refracted rays; alſo κ and k two other conjugate focuſes; and when the interval $\varpi\kappa$ of the focuſes of incident rays is exceeding ſmall, the interval pk of the focuſes of the refracted rays will be $\frac{mnrr}{\overline{\theta\varpi - mr}|^2} \times \varpi\kappa$; ſuppoſing m, n, θ to ſignify the ſame as in the former propoſition.

For let t be the focus of parallel rays coming the contrary way to the incident ones that flow from ϖ, and by art. 224, $ot = \frac{m}{\theta}\, or$ and $rt = \frac{n}{\theta}\, or$; and by art. 238, $\varpi t : \varpi r :: \varpi o : \varpi p$, and conjointly $\varpi t : tr :: \varpi o : po$, that is (for $or, o\varpi, ot, op$ putting r, ϖ, t, p) $\frac{m}{\theta} r - \varpi : \frac{n}{\theta} r :: \varpi : p = \frac{nr\varpi}{mr - \theta\varpi}$; and conſequently putting k and κ inſtead of p and ϖ, we have $k = \frac{nr\kappa}{mr - \theta\kappa}$; and ſo the diſtance $pk = (p - k =)\ \frac{mnrr \times \overline{\varpi - \kappa}}{\overline{mr - \theta\varpi} \times \overline{mr - \theta\kappa}} = \frac{mnrr}{\overline{mr - \theta\varpi}|^2}$

* Art. 650. $\times \varpi\kappa^{*} = \frac{mnrr}{\overline{\theta\varpi - mr}|^2} \times \varpi\kappa$, becauſe the ſquare of any quantity is the ſame whether its root be affirmative or negative. *Q. E. D.*

PROPOSITION II.

652. *Having the focus of homogeneal rays incident upon any lens, it is propoſed to find the aberrations of the refracted rays.*

Fig. 535, 536. Let $OIEo$ be the given lens, whoſe vertexes are O and o; R the center of the firſt ſurface OI; r the center of the ſecond oE; and in the axis $oOrR$

a Art. 211. let P be the focus of incident rays, and p the geometrical [a] focus of the refracted ones; it is propoſed to find pl the aberration of the ray $PIEl$. From the points of incidence and emergence I, E, draw IV, Ev perpendicular to the axis, and putting D for the difference of the thickneſſes Oo, Vv; P, R, r for the lines OP, OR, ϱr; and 2 to 3 for the ratio of refraction in glaſs, I ſay

the aberration
$$pl = \frac{\left.\begin{matrix}-27RR\\+6Rr\\-7rr\end{matrix}\right\}PP\;\left.\begin{matrix}+66RRr\\+14Rrr\end{matrix}\right\}P\;-52RRrr}{6PP\times\overline{R-r}|^2-24PRr\times\overline{R-r}+24RRrr}\,D,$$
ſuppoſing the lines P, R, r to lye all on one ſide of the glaſs; and when they have different poſitions in different glaſſes, their ſigns in the theorem muſt be altered accordingly.

For let ϖ be the conjugate focus to P after the firſt refraction at I, and Ix the firſt refracted ray; then calling VI and vE, I and E, and in cor. 1. prop. 1. for $\mathcal{Q}$, S, m, n, θ, Y put P, R, 3, 2, 1, I, and the firſt aberration $\varpi x = \overline{\frac{P-R}{P+2R}}\Big|^2 \times \frac{4P-10R}{6PR}\times II$. Again taking xIE for the incident ray upon the ſecond ſurface oE, let k be the conjugate focus to x after the ſecond refraction; then call ox and $O\varpi$, x and ϖ; and in cor. 1. prop. 1. for $\mathcal{Q}$, S, m, n, θ, Y put x, r, n, m, $-\theta$, E or I, (for they are equal, being in the ratio of xv to xV,) and the ſecond aberration $kl = \overline{\frac{x-r}{x-\frac{m}{\theta}r}}\Big|^2 \times \frac{mmr+mnr-mmx}{2\theta xxr}\times II = \overline{\frac{x-r}{x-3r}}\Big|^2 \times \frac{15r-9x}{4xr}II = \overline{\frac{\varpi-r}{\varpi-3r}}\Big|^2 \times \frac{15r-9\varpi}{4\varpi r}II$*. But p is the conjugate focus to ϖ, and ſo by the 2d lemma, $pk = \frac{6rr}{\overline{\varpi-3r}|^2}\times(\varpi x)\overline{\frac{P-R}{P+2R}}\Big|^2 \times \frac{4P-10R}{6PR}II$; and by art. 236, $\frac{3PR}{P+2R} = \varpi = \frac{3PR}{t}$, by putting t for $P+2R$. Subſtitute this value for ϖ; then $pk = \frac{8r^3\times\overline{P-R}^2\times\overline{2P-5R}}{\overline{RP-rt}|^2\times 36RrP}II$, and $kl = \frac{3\overline{RP-rt}|^2\times\overline{5rt-9RP}}{\overline{RP-rt}|^2\times 36RrP}II$. By adding pk and kl together and reſtoring the value of t and dividing all by 3, we have the total aberrration
$$pl = \frac{\left.\begin{matrix}-27R^3\\+33RRr\\-13Rrr\\+7r^3\end{matrix}\right\}PP\;\left.\begin{matrix}+66R^3r\\-52R^2r^2\\-14Rr^3\end{matrix}\right\}P\;\begin{matrix}-52R^3rr\\+52R^2r^3\end{matrix}}{12RrPP\times\overline{R-r}|^2-48RRrrP\times\overline{R-r}+48R^3r^3}\,II.$$
Now $OV = \frac{II}{2R}$ and $ov = \frac{II}{2r}$; whence $Vv = (OV+Oo-ov=)\ \frac{II}{2R}+Oo-\frac{II}{2r}$, and ſo $D = (Oo-Vv=)\ \frac{R-r}{2Rr}II$, and $II = \frac{2Rr}{R-r}D$. By putting this value inſtead of II and dividing every term by $R-r$, we have

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* Art. 650.

$$pl = \frac{\left.\begin{matrix}-27RR\\+6Rr\\-7rr\end{matrix}\right\} PP \left.\begin{matrix}+66RRr\\+14Rrr\end{matrix}\right\} P - 52RRrr}{6PP\times\overline{R-r}|^{2} - 24PRr\times\overline{R-r} + 24RRrr} D. \quad Q.E.D.$$

653. *Corol.* 1. When the incident ray is parallel to the axis, its aberration $pl = \frac{-27RR+6Rr-7rr}{12Rr\times\overline{R-r}} II$ or $= \frac{-27RR+6Rr-7rr}{6\times\overline{R-r}|^{2}} D$. For when P is infinite the terms of the general theorems which contain PP are incomparably greater than all the reſt.

Definition. Fig. 537. 654. Suppoſing a convex lens to have no thickneſs at its edges and a concave one to have none in the middle; when the ray falls upon the edge of either lens, then D becomes equal to the difference of the verſed ſines of the half arches that compoſe it. This I call the thickneſs of the lens, which being given we have the aberration of the extream ray by either of the theorems.

Fig. 538. 655. *Corol.* 2. The thickneſs D, focal diſtance p, and ſemidiameter R of the firſt ſurface, being given, the aberration of the extream ray coming parallel to the axis, that is $pl = \frac{27RR+24Rp+7pp}{6RR} D$, when the centers of the ſurfaces of the glaſs lye both on one ſide of it; but if they lye on contrary ſides, $pl = \frac{27RR-24Rp+7pp}{6RR} D$. For by art. 234, the focal diſtance $p = \frac{2Rr}{R-r}$, whence $r = \frac{Rp}{p+2R}$, which being put for r in the theorem of cor. 1, gives the preſent one.

656. *Corol.* 3. The thickneſs D, focal diſtance p and ſemidiameter r of the ſecond ſurface, being given, the aberration of the extream ray coming parallel to the axis, will be $pl = \frac{7pp+4rp+rr}{6rr} D$ when the centers of the ſurfaces lye both on the ſame ſide of the glaſs; but if they lye on contrary ſides, it will be $\frac{7pp-4rp+rr}{6rr} D$; found by putting $\frac{rp}{p-2r}$ for R in cor. 1.

657. *Corol.* 4. When the focus of incident rays upon a given lens is given, or when they are parallel, the aberrations of the rays from the conjugate focus are as, II, the ſquares of the diſtances of the points of incidence or emergence from the axis of the lens. For in this caſe P, R, r being given, pl is as II.

LEMMA II.

658. The focal diſtance, half the breadth, and the thickneſs of any lens, are continual proportionals.

At the latter end of the demonſtration of prop. 2. we had $II = \frac{2Rr}{R-r}D$, and by art. 232 the focal diſtance $p = \frac{2Rr}{R-r}$; whence $II = pD$, or p, I, D are continual proportionals. *Q. E. D.*

659. *Corol.* 1. Hence in glaſſes of all ſorts of ſhapes, (that is whatever be the ratio and poſition of the ſemidiameters of their ſurfaces,) if any two of theſe three things, focal diſtance, breadth and thickneſs, be the ſame, the third is the ſame alſo.

PROPOSITION III.

660. *To compare the aberrations cauſed by the ſphericalneſs of the figure of all ſorts of glaſſes: and to determine the ſemidiameters of a glaſs which ſhall make the leaſt aberrations.*

To make a juſt compariſon we muſt ſuppoſe all our glaſſes to have the ſame focal diſtance, the ſame breadth, and conſequently the ſame thickneſs by art. 659; and to differ only in their ſhapes, ariſing from the various magnitudes and poſitions of the ſemidiameters of their ſurfaces.

661. Firſt then, I ſay, that when parallel rays fall upon the plane ſide of a plano-convex glaſs, the aberration of the extream ray, which is $\frac{9}{2}$ of the thickneſs, is leſs than the like aberration cauſed by any meniſcus glaſs whoſe concave ſide is expoſed to the incident ray.

662. Secondly, I ſay that when the ſaid glaſſes have their convexities turned to the incident rays, the aberration of the extream ray in the plano-convex, which is now but $\frac{7}{6}$ of its thickneſs, is leſs than the like aberration of any meniſcus in this poſition.

663. Thirdly, I ſay that a double convex glaſs, when the ſemidiameter of the firſt ſurface, upon which the rays fall, is to that of the ſecond from whence they emerge as 2 to 5, is juſt as good as the plano-convex in its beſt poſition; the aberrations of both being $\frac{7}{6}$ of their common thickneſs.

664. Fourthly, I ſay that when the ſemidiameters of a double convex are equal, it is not ſo good as a plano-convex in its beſt poſition, its aberration being $\frac{5}{3}$ of its thickneſs: but if the ſemidiameters of its firſt and ſecond ſurfaces be as 1 to 6, it is the beſt glaſs of all; the aberration of the extream ray being now but $\frac{15}{14}$ths of its thickneſs; which is the leaſt poſſible; there being no ſuch thing in nature as a glaſs compoſed of two ſpherical ſurfaces that has no aberrations. But if this beſt glaſs be inverted it becomes much worſe; for the aberration will then be $\frac{145}{42}$ of its thickneſs.

665. Laſtly, I ſay that when a plano-concave has its plane ſide expoſed to parallel rays, the aberration of the extream ray is alſo $\frac{9}{2}$ of its thickneſs, and that when it is inverted, the aberration becomes only $\frac{7}{6}$ths; which is

leſs than the aberration of any concavo-convex glaſs; and equal to that of a double concave glaſs the ſemidiameter of whoſe firſt ſurface is to that of the ſecond as 2 to 5; and that the beſt of all double concave glaſſes has the ſemidiameters of its firſt and ſecond concavities as 1 to 6; and conſequently this is the beſt figure of a glaſs to help ſhort-ſighted perſons, as the double convex one of the like figure is the beſt for ſpectacles.

DEMONSTRATION.

666. Firſt then, by article 655. the aberration cauſed by a meniſcus is $\frac{9}{2}D + \frac{4p}{R}D + \frac{7}{6}\frac{pp}{RR}D$; which decreaſes continually as R increaſes, till when R is infinite and the meniſcus becomes a plano-convex, the aberration becomes equal to $\frac{9}{2}D$, the other terms being incomparably leſs than this.

667. Secondly, after R is infinite let it become negative; and after that let r alſo become infinite and then negative, though bigger than R, that R may belong to the firſt ſurface whoſe convexity is now expoſed to the incident rays; and by art. 656. the aberration, now cauſed by the meniſcus, namely $\frac{7}{6}\frac{pp}{rr}D + \frac{2p}{3r}D + \frac{7}{6}D$* decreaſes continually as r the ſemidiameter of its ſecond ſurface increaſes, till it equals $\frac{7}{6}D$, when the glaſs becomes a plano-convex: which is leſs than its aberration $\frac{9}{2}D$, in the contrary poſition, in the ratio of 7 to 27, or almoſt 4 times.

668. Thirdly, to change this glaſs into a double convex, after r is infinite let it become negative, and ſuppoſing it vaſtly bigger than p, the negative term $\frac{2p}{r}D$, in the preſent aberration $\frac{7}{6}\frac{pp}{rr}D - \frac{2p}{3r}D + \frac{7}{6}D$, will be bigger than the affirmative one $\frac{7}{6}\frac{pp}{rr}D$; (the root of any proper fraction being bigger than its ſquare in a duplicate ratio of the terms;) conſequently their negative difference will make the aberration of a double convex leſs than $\frac{7}{6}D$, the aberration of a plano-convex; ſo long as that difference continues negative; but ſuppoſing r of ſuch a magnitude as will make that difference nothing, or $\frac{7}{6}\frac{pp}{rr}D - \frac{2p}{3r}D = 0$, the aberration of this double convex will be $\frac{7}{6}D$, the ſame as that of the plano-convex. Now for p put its value $\frac{2Rr}{R-r}$ and you have $\frac{7}{3} \times \frac{R}{R-r} - \frac{2}{3} = 0$, whence $5R = 2r$, that is $R : r :: 2 : 5$.

669. Fourthly, the aberration of any double convex glaſs whoſe ſemidiameters are given may be found by computing this quantity $\frac{27RR + 6Rr + 7rr}{6 \times \overline{R+r}\,|^{2}}D = pl$ by art. 653, the ſign of r or R being changed, be-

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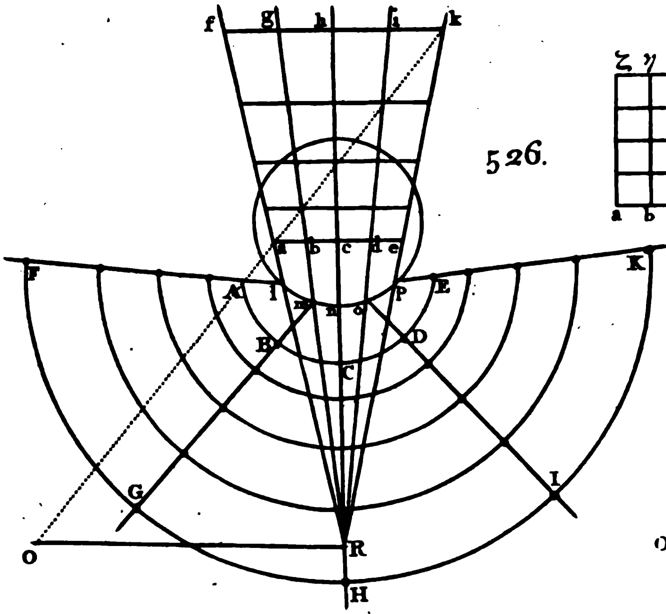
* See the table of Errata for p. 256.

cauſe



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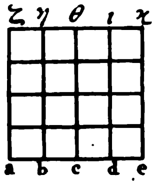

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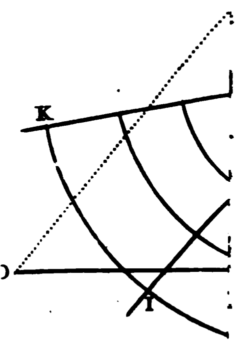

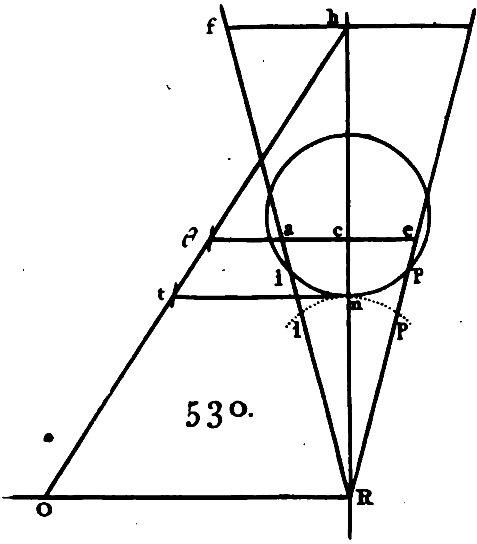

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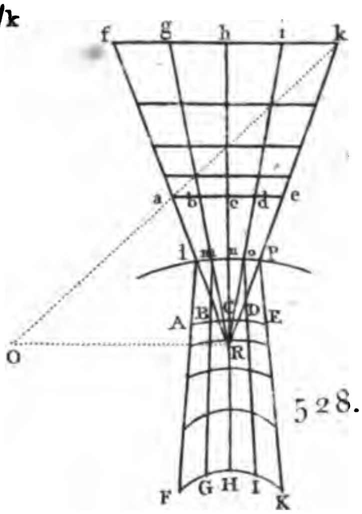

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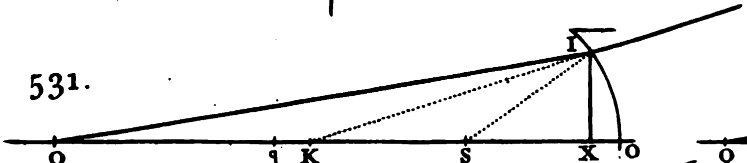

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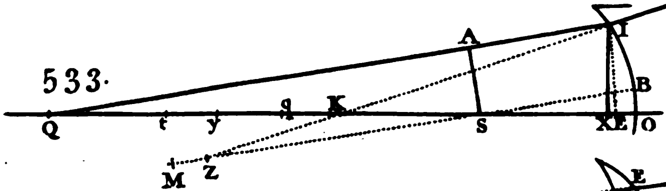

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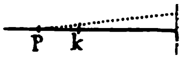

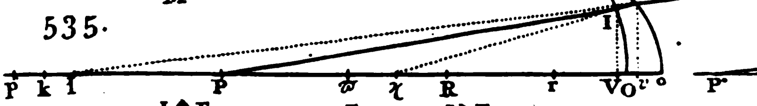

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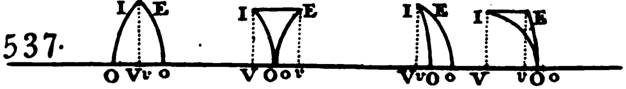

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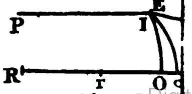

cauſe they now lye on contrary ſides of the glaſs. Thus ſuppoſing $R=2$, $r=5$, the aberration comes out $\frac{7}{6}D$, as was proved juſt now; and ſuppoſing $R=r$, it comes out $\frac{5}{3}D$. To find the ratio of R to r when the aberration is the leaſt poſſible, put $R=1$, then the aberration is $\frac{27+6r+7rr}{6\times\overline{1+2r+rr}}\times D=pl$. Now when any quantity or fraction comes to its leaſt (or greateſt) magnitude, it varies that magnitude but very little, or rather not at all while r varies but little. Suppoſe r increaſed by an exceeding ſmall increment n, and by ſubſtituting $r+n$ for r, $rr+2rn+nn$ for rr, we have $\frac{6}{D}pl=\frac{27+6r+7rr}{1+2r+rr}=\frac{27+6r+7rr+6n+14rn+7nn}{1+2r+rr+2n+2rn+nn}$. Which gives this proportion, $27+6r+7rr : 1+2r+rr :: 27+6r+7rr+6n+14rn+7nn : 1+2r+rr+2n+2rn+nn$ (and disjointly) $:: 6n+14rn+7nn : 2n+2rn+nn :: 3+7r : 1+r$, neglecting the ſquares of n as inconſiderable[a]; and alternately $27+6r+7rr : 3+7r :: (1+2r+rr : 1+r ::)\ 1+r : 1$. Hence $27+6r+7rr=3+10r+7rr$, and by reduction $r=6$. But R was put 1, and ſo $R : r :: 1 : 6$. Put theſe values for R and r, and you will find the aberration $pl=\frac{15}{14}D$. Now if this beſt glaſs be inverted, that is, if we put $R=6$ and $r=1$, you will find $pl=\frac{145}{42}D$.

[a] Art. 650.

670. Since this glaſs in its beſt poſition has the leaſt poſſible aberration, it is impoſſible for any ſpherical glaſs to have no aberration at all; which may alſo be proved after this manner. Suppoſing it poſſible that pl may be nothing, then putting $r=1$, we have $27\pm6r+7rr=0$ whence by reduction $r=\pm\frac{3}{7}\pm\sqrt{\frac{9}{49}-\frac{27}{7}}$, which is an impoſſible quantity; and ſo the ratio of R to r is impoſſible upon ſuppoſition that the aberration is nothing.

671. Laſtly it appears by art. 655 and 656, that thoſe theorems ſerve equally for concave and for convex glaſſes, and ſo the ſame demonſtrations ſerve alſo for this laſt article. Q. E. D.

LEMMA IV.

672. If the angles of incidence and refraction of a ray, *QACS*, that paſſes through a very ſmall angle of a priſm, *AIC*, be ſo little as to be reckoned proportionable to their ſines; the angle of deviation *RFS*, contained under the incident ray *QAFR* and the emergent ray *SCFT* produced, will be to the refracting angle *AIC*, as the difference of the ſines of incidence and refraction to the leſſer of them; and conſequently the magnitude of the angle of deviation *RFS* will be invariable in all poſitions of the ray. Fig. 539, 540.

For let the perpendicular *AB*, to the firſt ſurface *AI*, croſs *CD*, the perpendicular to the ſecond, in *E*; and ſuppoſing the ray *AC* to go both

ways

ways out of the priſm, the angle of incidence ACD will be to the angle of emergence DCT, in the given ratio of the ſine of incidence to the ſine of refraction, that is of i to r; and diſjointly, we have ACD to ACT as i to $r-i$; and the angle CAB is to CAR, in the ſame ratio, ſuppoſing the ray to go backward along CA; and conjointly or diſjointly we have $ACD \pm CAB$ to $ACT \pm CAR$, that is AED or AIC to RFS in the ſame given ratio of i to $r-i$. *Q. E. D.*

673. *Corol.* 1. Hence any two homogeneal emergent rays produced, will be inclined to one another in the ſame angle as the two incident rays Fig. 541. are inclined to one another. For let the two incident rays QF, qf (produced) meet in K; and let the emergent rays SF, sf (produced meet in L; and let one of the incident rays croſs the other emergent ray in M; and ſince, in the triangles KMF, LMf, the angles at M are equal and alſo thoſe at F and f by this lemma, it follows that the remaining angles at K and L are alſo equal.

674. *Corol.* 2. And when two homogeneal rays are refracted through the ſame point of any lens, whoſe thickneſs is inconſiderable, the angle contained under their incident parts is equal to the angle under their emer-Fig. 72, 73. gent parts. For the thickneſs of the lens being very ſmall, if the rays have a common point of incidence, their points of emergence will be very near one another; or if the point of emergence be common to both rays, their points of incidence will be very near one another; and ſtill nearer if the rays croſs one another within the lens; conſequently the refractions through the lens will be nearly the ſame as through two planes, that touch its ſurfaces at two given points, near the points of incidence and emergence and contain a given angle with one another.

Fig. 542. 675. *Corol.* 3. When the ray AC within the priſm, coincides with a perpendicular to either of the planes, as with AB; one of the refractions will vaniſh at A; and then the angle of deviation RFS made by the other ſingle refraction, will continue the ſame in quantity as before, when it was made by two refractions; becauſe the magnitude of the angle of deviation is invariable by this lemma.

676. *Corol.* 4. Therefore when an heterogeneal ray is ſeparated into coloured rays, by ſmall refractions through a ſmall refracting angle of a given quantity, the emergent rays of given colours will be inclined to one another and to the incident ray in certain given angles, in all poſitions of the incident ray. Becauſe theſe inclinations made by two refractions, are every where equal to the inclinations made by a ſingle refraction at the ſecond plane, when the incident ray falls perpendicular upon the firſt plane.

677. *Corol.* 5. And an heterogeneal ray, refracted through a given point in a lens, has the ſame property as in the priſm; that is the emergent rays

of given colours are inclined to one another and to the incident ray in given angles in all poſitions; for the reaſon mentioned in cor. 2.

678. *Corol.* 6. Therefore if ſeveral glaſſes of ſeveral ſorts or ſhapes have the ſame focal diſtance, and the ſame aperture; the diameter of the circle of aberrations of heterogeneal parallel rays from their principal focus, will be the ſame in them all; being the ſame as in a plano-convex glaſs when its plane ſide is turned to the incident rays[a]; and in this caſe we have determined it above in art. 324. And when the rays in the incident pencil are either parallel or inclined to the axis of the lens, the diameter of the circle of aberrations is as its diſtance from the lens; becauſe the angle RAS is invariable.

[a] Art. 675. 676.

Fig. 376.

679. *Corol.* 7. Therefore with reſpect to theſe aberrations by colours ſeparately conſidered, it is indifferent which ſide of a lens is turned to the incident rays; becauſe its focal diſtance is the ſame in both poſitions[b].

[b] Art. 233.

PROPOSITION IV.

680. *When* Q *the focus of homogeneal incident rays is not much farther from a lens* EI *than its focal diſtance* EF; *(as in double microſcopes;) the lateral aberration* qR *of the outermoſt refracted ray* IR, *from their geometrical focus* q, *is to* FG *the lateral aberration (from the principal focus) of a ray* PI *that comes the contrary way, parallel to the axis and through the ſame point* I, *as* Eq *to* EF *nearly; that is directly as the diſtances of theſe focuſes of refracted rays from the lens.*

Produce the perpendicular GF till it cuts the incident ray QI in L; and joining LE and Iq they will be parallel. For the triangles QLF, QIE being ſimilar, we have $QL : QI :: (QF : QE ::) QE : Qq$*; which ſhews that the triangles QLE, QIq are equiangular. Draw EM parallel to the refracted ray IKR cutting FL produced in M; then the angle GIL equals PIK* or IKE or MEF; and conſequently GL equals FM nearly; being the ſubtenſes of the equal angles GIL, MEF and being very ſmall and nearly perpendicular to their legs when QF and EI are very ſmall. Hence taking away the common line FL, we have LM equal to the aberration FG. But the angles qIR, LEM, under two couple of parallel lines, are equal; and their ſubtenſes qR, LM being equally inclined to their legs, we have $qR : LM$ or $FG :: (qI : EL ::) qE : EF$; becauſe the triangles qIE, ELF are equiangular. *Q. E. D.*

Fig. 545.

* Art. 239.

* Art. 674.

681. *Corol.* 1. When a plano-convex glaſs, which is nearly as good as the beſt of all[c], has its plane ſide turned towards the focus Q, we have the lateral aberration cauſed by its figure, that is $qR = \frac{7}{6} \times \frac{EI^3}{EF^3} \times Eq$, and the longitudinal aberration $qK = \frac{7}{6} \times \frac{EI^2 \times Eq^2}{EF^3}$: and in ſpherical lenſes of any ſhape

[c] Art. 662. 664.

ſhape or ſubſtance, their aberrations are as theſe quantities. For the longitudinal aberration of the parallel ray PI, that is Ff is $\frac{7}{6}$ of the thickneſs of the glaſs[a] or $\frac{7}{6} \times \frac{EI^2}{EF}$ *; and becauſe the triangles fFG, fEI are equiangular, we have the lateral aberration $FG = \frac{7}{6} \times \frac{EI^3}{EF^2}$; and therefore by this propoſition we have $qR = \frac{7}{6} \times \frac{EI^3}{EF^3} Eq$; and becauſe the triangles qRK, EIK are equiangular, we have $qK = \frac{7}{6} \times \frac{EI^2 \times Eq^2}{EF^3}$.

[a] Art. 662.
* Art. 658.

682. *Corol.* 2. Hence FQ being given, the lateral aberrations, cauſed by the figure of any given lens, are as the cubes, and the longitudinal aberrations as the ſquares, of the diameters of the apertures.

683. *Corol.* 3. And hence the diameter of a circle of theſe aberrations cauſed by the figure only, that is $\frac{1}{2} qR$* $= \frac{7}{12} \times \frac{EI^3}{EF^3} \times Eq$. Becauſe the demonſtration of the 339th article has this foundation only, that the longitudinal aberrations are as the ſquares, and the lateral ones as the cubes, of the linear apertures; and conſequently ſerves for any lens by cor. 2. as well as for a plano-convex one.

* Art. 339.

684. *Corol.* 4. The diameter of a circle of aberrations that will juſt contain all ſorts of heterogeneal rays flowing from Q, is equal to $\frac{2}{55} \times \frac{Eq}{EF} EI$, by art. 324 and 678, taking F and q for the focuſes of the mean refrangible rays.

685. *Corol.* 5. Therefore when the plane ſide of the object-glaſs EI, of a double microſcope, is turned towards the object at Q, the diameter of the circle of aberrations in its image at q, that would ariſe in homogeneal rays from the ſpherical figure only, would be to the diameter of a circle of aberrations of heterogeneal rays, if the figure of the lens was perfect, as 1 to $\frac{24}{385} \times \frac{EF^2}{EI^2}$. And conſequently theſe circles will be equal when the diameter of the aperture of the object-glaſs is half its focal diſtance very nearly.

LEMMA V.

686. Let innumerable pencils of rays be ſuppoſed to belong to innumerable focuſes in any given part of the axis of the eye produced both ways; and let it be propoſed to determine the leaſt circle upon the retina, into which all the rays, that enter the pupil, can be collected.

Fig. 544 to 547. Let AO be the ſemidiameter of the pupil, and OTS the axis of the eye produced; and let pencils of rays belong to every point of the part ST, when its extremities S, T are both on the ſame ſide of the eye; or elſe to every point of the infinite axis, excepting the part ST, when S and T are on contrary ſides of the eye. Biſect ST in V, and towards the eye take

VQ to VT as VT to VO; and through Q draw a line PQR perpendicular to the axis, cutting SA and TA, the outermoſt rays of the extream pencils, in P and R reſpectively. And when the eye adapts it ſelf to view the line PR as diſtinctly as poſſibly, its picture $\varpi\rho$ upon the retina will be the diameter of the leaſt circle into which all the rays can be collected.

For ſince we made $VO : VT$ or $VS :: VS : VQ$, conjointly we have $VO + VS$ to $VS + VQ$ in the ſame ratio; and alſo disjointly $VO - VT$ to $VT - VQ$ in the ſame ratio; that is $SO : SQ :: TO : TQ$. Conſequently, ſince the triangles SAO, SPQ and alſo TAO, TRQ are equiangular, we have $AO : PQ :: (SO : SQ :: TO : TQ ::)\ AO : QR$, and therefore the lines QP, QR are equal. Let $\varpi\kappa\rho$ be a diſtinct picture of PQR, that is let all the rays that may be ſuppoſed to flow from P, be collected to ϖ upon the retina; and the ray SPA being one of them, will go to ϖ along with the reſt; and in like manner the ray RTA will go to ρ; and the lines PQ, QR being equal, their pictures $\varpi\kappa$, $\kappa\varrho$ will alſo be equal. Imagine the whole figure to be turned about the axis STO; and it will appear that the rays of one extream pencil, whoſe focus is S, will be ſcattered all over a circle upon the retina, whoſe center is κ and ſemidiameter is $\varpi\kappa$; and likewiſe the rays of the other extream pencil, whoſe focus is T, will be ſcattered all over a concentrick circle, whoſe ſemidiameter is $\kappa\varrho$; and theſe two circles will exactly coincide when their ſemidiameters $\kappa\varpi$, $\kappa\varrho$ are equal. But if the object PQR be ſuppoſed to approach toward the eye, the ſemidiameters PQ and $\varpi\kappa$ will both increaſe; and on the other hand, if PQR be ſuppoſed to recede from the eye, the ſemidiameters QR and $\kappa\varrho$ will both increaſe, while the other ſemidiameters PQ and $\varpi\kappa$ are decreaſing; and in both ſuppoſitions the circle upon the retina which contains the rays of both the extream pencils, being the larger of the two, will be larger than it was before, when it was equal to the other; that is when the eye viewed the equal lines PQ, QR diſtinctly.

Now when S and T are on the ſame ſide of the eye, all the pencils that belong to intermediate focuſes between the extreams S, T will fall within the circle above defined. For conceiving the neareſt point T to recede from the eye, the lines QR and $\kappa\varrho$ will both decreaſe, and ſo will QP and $\kappa\varpi$ while S approaches towards the eye. And if S and T be on contrary ſides of the eye, all the pencils will be collected into the ſaid circle, whoſe focuſes are in every point of the infinite axis excepting the part ST. For by conceiving both OS and OT to increaſe, QP and QR, and conſequently $\kappa\varpi$ and $\kappa\varrho$ will both decreaſe. *Q. E. D.*

687. *Corol.* 1. Hence when a pencil of rays that flow from a ſingle point of an object, ſhall be ſo diſpoſed by reflections and refractions, as in falling upon the eye, to belong to innumerable different points of its

axis produced, as in the propoſition; the diameter $\varpi\varrho$ of the leaſt circle upon the retina, that contains them all, will be as the angle SAT; that is as the greateſt angle in which the two outermoſt rays interſect one another at the pupil of the eye. For $\varpi\varrho$ is as the angle POR or PAR or SAT.

Fig. 546. 688. *Corol.* 2. Conſequently when OS and OT are on contrary ſides of the eye, and are equal to one another; that is when the two outermoſt rays are equally inclined to the axis and contrary ways, the diameter $\varpi\varrho$ of the leaſt circle of aberrations, will be as the angle AST or ATS in which either of the outermoſt rays is inclined to the axis: and in this caſe the eye muſt be adapted to collect the parallel rays to a diſtinct point upon the retina. For the line VO being nothing in this caſe, the third proportional VQ becomes infinite; and $\varpi\varrho$ being always as the angle POR or PAR is now as its half, AST or ATS.

DEFINITION.

689. In viſion either with the naked eye or with glaſſes, the apparent indiſtinctneſs of a given object, is as the area of the leaſt circle upon the retina, into which all the rays of a ſingle pencil can be collected by the eye.

The reaſon of this definition has been ſhewn in the beginning of the demonſtration of the 343d article.

PROPOSITION V.

690. *In microſcopes made with ſingle lenſes, a given object placed at their principal focuſes will appear equally diſtinct, if their linear apertures be as their focal diſtances.*

Fig. 548. [a] Art. 81. 691. *Caſe* 1. At firſt let us ſuppoſe the figure of the lens DP, to be ſuch as would cauſe no aberrations of homogeneal rays[a]; then if the object be placed at F, and FP be the focal diſtance of mean refrangible rays, all theſe rays that flow from F, will be refracted into the lines DE parallel to the axis FPQ. Let the violet contained in the outermoſt heterogeneal ray FD be refracted along DK; and let another violet ray be ſuppoſed to come backwards along ED, and to be refracted along DR, meeting the object in R. Then the apparent indiſtinctneſs of the point F being as the area of the leaſt circle of aberrations, upon the retina, of the rays in one pencil flowing from F*, will be given when the diameter of this circle is given; or when the angle EDK* or FDR* is given; that is when the ratio of FR to FD or FP is given. But in heterogeneal rays the ratio of FR to DP is alſo given[b]; and conſequently the apparent indiſtinctneſs will be given when the ratio of DP to PF is given.

* Art. 689.
* Art. 688.
* Art. 674.
[b] Art. 324. 678.

692. *Caſe* 2. Now let the figure of the lens *DP* become ſpherical while it keeps the ſame focal diſtance; and the innermoſt of the mean refrangible rays flowing from *F*, will ſtill be refracted into lines parallel to the axis; but the outermoſt of them as *DE* will now converge towards it by too great a refraction[a]. Imagine a mean refrangible ray to come backwards along *LD* parallel to the axis, and to be refracted along *Dr*, ſo that *Fr* may be the lateral aberration of homogeneal rays cauſed by the figure of the lens. And the apparent indiſtinctneſs of the object ſo far as it depends upon this ſort of aberration, will be given, as in caſe 1, when the ratio of *Fr* to *FP*, that is, of $\frac{DP^3}{FP^2}$* to *FP*, and conſequently of DP^3 to FP^3, and of *DP* to *PF*, is given, as before in caſe 1.

Fig. 549.

[a] Art. 81.

* Art. 681.

693. *Caſe* 3. Now let the violet contained in the heterogeneal ray *FD* be refracted along *DK*; and the angle *EDK* between the green and violet will be a little bigger than in the firſt caſe; being increaſed by a 27th or 28th part of the angle *EDL**, which meaſures the increaſe of refraction or deviation cauſed by the change of the figure of the lens. This increment of *EDK* is therefore as the angle *EDL*, and conſequently is given when the ratio of *DP* to *PF* is given, by caſe 1. But the apparent indiſtinctneſs will be given, when the diameter of the leaſt circle of aberrations of all the rays upon the retina is given; that is when the whole angle *KDL* is given[b]; or when all its parts *EDL*, *EDK* and the increment of *EDK*, are given; and we have ſhewn that all theſe parts are given when the ratio of *DP* to *PF* is given. *Q. E. D.*

Fig. 549.

* Art. 324.

[b] Art. 688.

694. *Caſe* 4. In theſe caſes I have conſidered the aberrations of the violet rays from the green or mean refrangible ones, which were ſuppoſed to be parallel to the axis; and the red rays being inclined to the axis very nearly as much as the violet, and in a contrary poſition, will all be contained in the ſame circle of aberrations upon the retina[c]. *Q. E. D.*

[c] Art. 688.

695. In microſcopical lenſes whoſe focal diſtances are not much longer than half an inch, there is no need to contract their apertures, for procuring diſtinct viſion; the pupil it ſelf being ſmall enough to exclude the exterior ſtragling rays. But in ſmaller lenſes where apertures are neceſſary, we have ſhewn that to preſerve the ſame degree of diſtinctneſs their diameters muſt be as their focal diſtances; and then the apparent brightneſs will decreaſe in a duplicate ratio of their focal diſtances, ſo that by uſing ſmaller glaſſes the apparent magnitude and the obſcurity of the object will both increaſe in the ſame ratio. For the ratio of *PD* to *PF* being invariable, the angle *PFD* is alſo invariable, and conſequently the quantity of light received from the point *F* is alſo invariable; becauſe the apertures of the lenſes whether ſmaller or larger muſt all be ſituated at ſuch diſtances from *F*, as juſt to receive all the rays contained in a cone deſcribed by turning the angle *PFD* about the axis *PF*, neither more

Fig. 549.

nor leſs. But the apparent magnitude of the object, or the ſurface of its picture upon the retina is reciprocally as PF ſquare[a], and conſequently the light being the ſame its brightneſs is directly as PF ſquare. By this theory it appears that a minute object cannot be magnified to infinity by a ſingle lens though it were poſſible to make it as ſmall as we pleaſe; without ſome method of increaſing its light. Nevertheleſs this imperfection in ſingle microſcopes is not ſo great as at firſt ſight one would take it to be, or as in fact we find it; the reaſon may be becauſe the eye is capable of diſcerning objects tolerably well by above 20 thouſand different degrees of light, every degree being equal to the light by which we ſee objects in the brighteſt moon-light night[b]. But though the brightneſs of the object were increaſed by throwing new light upon it, yet *Huygens* obſerves that the power of the microſcope would ſtill be limited by the breadths of the pencils that enter the pupil, which is equal to the breadth of the aperture. For if this breadth be leſs than $\frac{1}{7}$ or $\frac{1}{6}$ of a line, he affirms that the edges of the object will begin to appear indiſtinct[c]. But by double microſcopes this excellent Author has made it appear we may magnify objects at pleaſure; provided it was poſſible to form their object-glaſſes as ſmall as we pleaſe; for we ſhall ſhew in the ſequel that all other obſtructions may be removed.

[a] Art. 118.

[b] Art. 95.

[c] Art. 358.

PROPOSITION VI.

696. *In refracting and reflecting microſcopes, and teleſcopes, made with a ſingle eye-glaſs, the apparent indiſtinctneſs of a given object, cauſed by the aberrations of either kind conſidered ſeparately, will be directly as the ſquare of the greateſt lateral aberration in the image formed by the object-glaſs or object-metal, and inverſely as the ſquare of the focal diſtance of the eye-glaſs, very nearly; becauſe the aberrations cauſed by the eye-glaſs are almoſt inconſiderable.*

Fig. 550, 551.

697. *Caſe* 1. Let us ſuppoſe the figures of the convex glaſſes EI, ei, to be perfect, or ſuch as would cauſe no aberrations of homogeneal rays[d]; and let all the mean refrangible rays that flow from the point Q, in the axis $QEqex$, be refracted through the object-glaſs EI to the focus q; where croſſing one another let them be refracted through the eye-glaſs ei into parallel lines; and let the humours of the eye be adapted to collect them accurately to a ſingle point x upon the retina $x\rho$. Then let the ray IRi be the outmoſt violet, or the outmoſt red, that was contained in the heterogeneal ray QI before refraction at I; and let this ray Ii be refracted by the eye-glaſs into the line ia, and be ſent by the refractions in the eye to the point ρ upon the retina. Laſtly imagine a ray bi of the ſame colour as ai to come backwards parallel to the axis, and to be refracted at i along the line ir; and let a perpendicular to the axis through q, cut the ray ir

[d] Art. 81.

in r and the ray iI in R Now the ſemidiameter $\kappa\rho$, of the circle of aberrations upon the retina, is as the angle aib* or as its equal Rir* and conſequently as $\frac{Rr}{Ri}$* or nearly as $\frac{Rq}{qe}$. For I ſhall ſhew preſently that qr is almoſt inconſiderable in compariſon to qR in all optical inſtruments, notwithſtanding it is not as qR. But the apparent indiſtinctneſs of the point Q, is as the area of the circle of aberrations upon the retina[a], or as the ſquare of its ſemidiameter $\kappa\rho$ or as $\frac{qR^2}{qe^2}$; that is directly as the ſquare of the lateral aberration in the image at q, and inverſely as the ſquare of the focal diſtance of the eye-glaſs. * Art. 688. * Art. 674. * Art. 222. [a] Art. 689.

698. *Caſe* 2. Suppoſing the glaſſes EI, ei to remain in the ſame places as before, let us now conſider the aberrations which the ſphericalneſs of their figures would cauſe in a pencil of mean refrangible rays. Therefore now let qR and $\kappa\rho$ repreſent the lateral aberrations of the outermoſt of theſe rays $QIRia\rho$; and let bir be alſo a mean refrangible ray. Then $\kappa\rho$ is as the angle aib* or as Rir* or as $\frac{Rr}{Ri}$* or as $\frac{Rq}{qe}$ neglecting qr. But the apparent indiſtinctneſs is as the circle of theſe aberrations at κ*, that is as $\kappa\rho^2$ or as $\frac{qR^2}{qe^2}$, that is, directly as the ſquare of the lateral aberration in the image at q and inverſely as the ſquare of the focal diſtance of the eye-glaſs; as in the former caſe. Fig. 550. * Art. 688. * Art. 674. * Art. 222. * Art. 689.

699. To ſhew that qr, the aberration by the eye-glaſs, may be neglected, let EF be the focal diſtance of the object-glaſs of the double microſcope; and let the ray Ii croſs the axis in K; then in caſe 1. we have $qR = \frac{1}{ss}\frac{Eq}{EF}EI$*, and $qr = \frac{1}{ss}ei = \frac{1}{ss}\frac{eq}{Eq}EI$; becauſe $ei : EI :: eK : KE :: eq : qE$ nearly. Conſequently we have $qR : qr :: \frac{Eq^2}{EF} : eq$; that is as the apparent magnitude of the object Q ſeen in the microſcope, to its apparent magnitude ſeen by the naked eye from a diſtance equal to Qq; as will appear by article 127. which gives ſome idea of the ratio of qR to qr. In *Huygens*'s ſtandard microſcope, to be deſcribed in art. 710, this ratio is as 35 to 1; and in a teleſcope it is alſo the ratio of its magnifying power. Fig. 550, 551. * Art. 678.

700. In caſe 2. conſidering the aberrations by the figure, we have $qR = \frac{7}{6}\frac{EI^3}{EF^3}Eq$* and $qr = \frac{7}{6}\frac{ei^3}{eq^2} = \frac{7}{6}\frac{EI^3}{Eq^3}eq$; for the reaſon abovementioned. Conſequently when the glaſſes E, e have ſimilar ſhapes, we have $qR : qr :: \frac{Eq^4}{EF^3} : eq :: 3500 : 1$ in *Huygens*'s microſcope. * Art. 681.

701. *Caſe* 3. Let us now conſider reflecting microſcopes and teleſcopes, in which let Q be the focus of incident rays upon a concave metal EI, whoſe ſemidiameter is EC and principal focus F; and let q be the conju- Fig. 552.

gate

gate focus to $\mathcal{Q}$; and $IKRe$ be a reflected ray making the aberration qK in length and qR in breadth; and let every thing elſe remain as in caſe 2. Then if we neglect the aberration qr cauſed by the refraction of the rays through the eye-glaſs ei; it is evident that the apparent indiſtinctneſs will be as $\frac{qR^2}{qe^2}$, as before in the refracting inſtruments.

702. To compare theſe aberrations qR and qr; we have the longitudinal aberration $qK = \frac{EI^2 \times qC^2}{EC^3}$. For by art. 648, we have $qK : \frac{1}{4}\frac{EI^2}{EC} :: qC^2 : CF^2$ or $\frac{1}{4}EC^2$. Hence the lateral aberrration $qR = \frac{EI^3}{EC^3} \times \frac{qC^2}{qE}$. Now in the eye-glaſs, if we conſider the aberrations cauſed by coloured rays, we had $qr = \frac{1}{55}\frac{eq}{qE} \times EI$[*], and by conſequence $qR : qr :: \frac{EI^2 \times qC^2}{EC^3} : \frac{1}{55} eq$; which in a reflecting microſcope hereafter deſcribed[a] is as 53 to 1. By ſuppoſing $E\mathcal{Q}$ to be infinitely increaſed, this microſcope will be changed into Sir *Iſaac Newton*'s reflecting teleſcope; and then we have $qR : qr :: \frac{EI^2 \times EF^2}{EC^3}$ or $\frac{EI^2}{8EF} : \frac{1}{55} eq :: 55 : 24$, in Mr. *Hadley*'s 5 foot teleſcope taking the middle eye-glaſs and aperture[b]. Nevertheleſs experience ſhews that the object appears ſufficiently diſtinct. Indeed the diſproportion of qR to qr will be greater if we neglect the darker and fainter colours into which the ray ia is ſeparated, which may ſcarce affect the eye, and take $qr = \frac{1}{250} ei$, ſuppoſing qe the focal diſtance of the brighteſt yellow[c]. But in refracting inſtruments the ratio of qR to qr will not be altered thereby[d]. With reſpect to the aberrations cauſed by the ſpherical figure of the eye-glaſs, we have $qr = \frac{7}{6}\frac{EI^3}{Eq^3} eq$[*], and by conſequence $qR : qr :: \frac{qC^2 \times qE^2}{EC^3} : \frac{7}{6} qe :: 7695 : 1$, in the reflecting microſcope hereafter deſcribed. In a reflecting teleſcope this ratio is compounded of the ratio of its magnifying power and of 1 to $9\frac{1}{3}$ the eye-glaſs being plano-convex.

[*] Art. 699.

[a] Art. 738.

[b] Art. 362.

[*] Art. 700.

[c] Newt. Opt. p. 86.

[d] Art. 699.

[*] Art. 70c.

703. *Corol.* Therefore in refracting and reflecting teleſcopes and double microſcopes, the apparent indiſtinctneſs of a given object, that would ariſe from the aberrations of either kind ſeparately conſidered, will be nearly the ſame or invariable, when the focal diſtances of their eye-glaſſes are as the greateſt lateral aberration in the images formed by their object-glaſſes or object-metals: or when the angle ſubtended by qR at the point i or e is invariable.

Definition. 704. And this angle is called the Angle of Aberration by the Figure or by Colours according as qR is the lateral aberration cauſed by the ſphericalneſs of the figure or by coloured rays.

The following propoſitions being wrote when I read over Mr. *Huygens* upon the ſame ſubject[a], I followed the letters of his figures, and cannot eaſily change them for ſuch as I commonly uſe without ſome danger of miſtakes.

LEMMA VI.

Concerning the apparent magnitude, the apparent brightneſs, and the angles of aberration, in double microſcopes compoſed of two convex glaſſes.

Fig. 553. 705. Let an object BX be placed a little farther from the object-glaſs PD than its principal focus O; and let its image NY be viewed through an eye-glaſs EZ whoſe focal diſtance is NE. Let the object BX be viewed alſo by the naked eye from any given diſtance $B\Omega$; then take $B\mathcal{Q}$ to BO as EN to OP, and the object will be magnified in the microſcope in the ratio of $B\Omega$ to $B\mathcal{Q}$.

For let the ray $XPYZ$ cut the image in Y, and be refracted by the eye-glaſs along ZV; join $X\Omega$ and draw $X\mathcal{Q}$ parallel to ZV or EY* and the object will appear from V under the angle EVZ equal to NEY or $B\mathcal{Q}X$; and conſequently will be magnified in the ratio of the angle $B\mathcal{Q}X$ to $B\Omega X$, or of $B\Omega$ to $B\mathcal{Q}$. But ſince the figures $PX\mathcal{Q}$, PYE are ſimilar, we have $B\mathcal{Q} : NE :: (BX : NY :: BP : PN ::) BO : OP$. For ſince B and N are conjugate focuſes, we have $BO : BP :: BP : BN$* and disjointly $BO : OP :: BP : PN$. Q. E. D.

706. *Corol.* 1. The apparent diſtance $B\mathcal{Q} = \frac{BP}{PN} NE = \frac{BO}{OP} NE$. And $B\Omega$ being given, the apparent magnitude of the object in the microſcope, is reciprocally as $B\mathcal{Q}$, by the lemma.

707. *Corol.* 2. Let PD be the ſemidiameter of the aperture of the object-glaſs, and the apparent brightneſs of the ſame object ſeen in the ſame or in different microſcopes, will be as $\frac{PD^2 \times NE^2}{PN^2}$. For the quantity of rays that illuminate any particle of its picture upon the retina, is as $\frac{PD^2}{PB^2}$; becauſe if PB was given, the quantity of rays received from B upon the whole aperture, would be as the aperture or as PD^2; and if PD was given, the quantity of thoſe rays would be as their denſity in the aperture, that is as $\frac{1}{PB^2}$*; but the apparent brightneſs of an object is directly as the quantity of rays that illuminate each particle of its picture upon the retina, and inverſely as the area of the picture, or inverſely as the apparent magnitude of the viſible area of the object; and ſo the apparent brightneſs is directly as $\frac{PD^2}{PB^2} \times \frac{PB^2 \times NE^2}{PN^2}$*.

[a] Dioptr. p. 233.

* Art. 50.

* Art. 239.

* Art. 58.

* Art. 705.

708. *Corol.* 3. The angle of aberration by colours is as $\frac{PD}{NE} \times \frac{PN}{PO}$ or as $\frac{PD}{NE} \times \frac{BP}{BO}$. For the greateſt lateral aberration by colours in the image at N is as $PD \times \frac{PN}{PO}$ *.

* Art. 684.

709. *Corol.* 4. The angle of aberration by the figure of the object-glaſs is as $\frac{PD^3}{PO^3} \times \frac{PN}{NE}$. For the greateſt lateral aberration by the figure in the image at N is as $\frac{PD^3}{PO^3}\, PN$ [a].

a Art. 683.

PROPOSITION VII.

Fig. 553, 554.

710. *To make a new refracting microſcope denoted by ſmall letters* enpdbo, *that ſhall magnify an object more than a given microſcope denoted by larger letters* ENPDBO, *in any propoſed ratio of* n *to* 1; *with the ſame degree of brightneſs, and diſtinctneſs too, ſo far as it depends upon the different refrangibility of rays and not upon the ſphericalneſs of the figure of the object-glaſſes.*

Take $ne = \frac{1}{n} NE$, $pd = \frac{1}{n} PD$; $po = \frac{1}{nn} PO$, $pb = \frac{1}{nn} PB$, $pn = \frac{1}{nn} PN$; and we have the microſcope required. For example, the dimenſions of *Huygens*'s ſtandard microſcope in inches were theſe; $NE = 2$, $PD = \frac{1}{20}$, $PO = \frac{7}{10}$, $PB = \frac{7}{9}$, $PN = 7$. And therefore the diameter of an object appeared in it 36 times longer than to the naked eye at 8 inches diſtance[b]. Now to find the dimenſions of another microſcope that ſhall magnify twice as much, we have $n = 2$. Whence by the rule $ne = \frac{1}{2} NE = 1$, $pd = \frac{1}{40}$, $po = \frac{7}{40}$, $pb = \frac{7}{36}$, $pn = \frac{7}{4}$.

b Art. 705.

The rule is grounded upon this Hypotheſis, of keeping the intervals of the points B, O, P, N, belonging to the object-glaſs, in the ſame given ratios. Hence we have NE reciprocally as the apparent magnitude [c] that is $ne : NE :: 1 : n$. Therefore $ne = \frac{1}{n} NE$. And becauſe the angle of aberrations by colours muſt remain unaltered[d], we have alſo PD as NE* or reciprocally as the apparent magnitude as before; that is $pd : PD :: 1 : n$. Therefore $pd = \frac{1}{n} PD$. And laſtly becauſe the apparent brightneſs muſt be kept unaltered, we have PN as $PD \times NE$* or as NE^2, (becauſe we had PD as NE,) or reciprocally in a duplicate ratio of the apparent magnitude; that is $pn : PN$ and $pb : PB$* and $po : PO$* :: $1 : nn$. Therefore $po = \frac{1}{nn} PO$; $pb = \frac{1}{nn} PB$ and $pn = \frac{1}{nn} PN$. Q. E. D.

c Art. 706.
d Art. 703.
* Art. 708.
* Art. 707.
* By Hypoth.

711. *Corol.* 1. In theſe microſcopes the angles of aberration by the figure of their object-glaſſes are directly in a duplicate ratio of the appa-

rent magnitudes of the object. For theſe angles are as $\frac{PD^3}{PO^3} \times \frac{PN}{NE}$ * or as * Art. 709. $\frac{PD^2}{PO^2}$ by the hypotheſis of the rule and becauſe we had PD as NE; that is, the angle of this aberration in the new microſcope, is to the like angle in the old, as $\frac{pd^2}{po^2}$ or $\frac{nnPD^2}{PO^2}$ to $\frac{PD^2}{PO^2}$, that is, as nn to 1.

712. *Corol.* 2. Hence if a given microſcope that has a plano-convex object-glaſs, will bear the convex ſide to be turned towards the object, which increaſes the angle of aberration by its figure near 4 times[a]; by [a] Art. 661. 662. 681. ſubſtituting for it a new plano-convex glaſs, with its plane ſide toward the object, whoſe focal diſtance is 4 times ſhorter, the angle of aberration will alſo become quadruple[b], becauſe the apparent magnitude of the object will [b] Art. 711. be doubled[c]. [c] Art. 710. Mr. *Huygens* found that his microſcope would bear this inverſion. But if we try to magnify much more by this propoſition, the aberrations by the figure will ſtill increaſe and put a ſtop to this proceſs; which nevertheleſs might be continued to infinity by the following propoſition, as this excellent Author has obſerved, but for the practical difficulty of making the object-glaſſes ſo ſmall as are requiſite for that purpoſe.

PROPOSITION VIII.

713. *To make a new refracting microſcope denoted by ſmall letters* enpdbo, *which ſhall magnify an object more than a given microſcope denoted by larger letters* ENPDBO, *in any propoſed ratio of* n *to* 1, *with the ſame brightneſs, and diſtinctneſs too, with reſpect to the aberrations cauſed by the figure; and with greater diſtinctneſs with reſpect to the aberrations cauſed by colours.* Fig. 553, 554.

Take $ne = \frac{1}{n} NE$, $pd = \frac{1}{n^3} PD$, $po = \frac{1}{n^4} PO$, $pb = \frac{1}{n^4} PB$, $pn = \frac{1}{n^4} PN$; and you have the dimenſions for the purpoſe required; and the angle of aberration cauſed by colours will be leſs in this than in the given microſcope, in the ratio of 1 to nn. For example in the ſtandard microſcope abovementioned, putting $n = 2$, we have $ne = 1$, $pd = \frac{1}{160}$, $po = \frac{1}{160}$, $pb = \frac{1}{144}$, $pn = \frac{1}{16}$; and the angle of aberration by colours 4 times leſs than before.

This rule is grounded upon the ſame hypotheſis as the former, *viz.* that the ratios of the intervals of the points B, O, P, N, are the ſame in both microſcopes. Therefore ſince the angles of aberration by the figure muſt be the ſame in both[d], we have PD^3 as $PO^2 \times NE$*; and ſince the [d] Art. 703. brightneſs muſt be the ſame too, * Art. 709. we have PN^2 or PO^2* as $PD^2 \times NE^2$*, * Hypoth. and conſequently PD^3 as $\frac{PO^3}{NE^3}$. * Art. 707. Subſtitute theſe values of PO^2

and PD^3 in the foregoing ſyſtem (PD^3 as $PO^2 \times NE$) and we have PD as NE^3, and PO as NE^4. But NE being reciprocally as the apparent magnitude*, or $ne : NE :: 1 : n$, we have $ne = \frac{1}{n} NE$; and PD being as NE^3, or $pd : PD :: ne^3 : NE^3 :: 1 : n^3$, we have $pd = \frac{1}{n^3} PD$; and in like manner PO being as NE^4, we have $po = \frac{1}{n^4} PO$; and by the hypotheſis pb and pn are as po.

Now the angle of aberration by colours is as $\frac{PD}{NE}$*; that is this angle in the new microſcope is to the like angle in the old, as $\frac{pd}{ne}$ or $\frac{PD}{nnNE}$ to $\frac{PD}{NE}$, that is, as 1 to nn. Q. E. D.

714. *Corol.* The breadth of the pencils that enter the pupil are alſo the ſame*, for half this breadth is $EI = \frac{PD \times NE}{PN}$; ſo that by this propoſition we might magnify to infinity without any impediments, but from the minuteneſs of the object-glaſs. But by the following propoſitions which do the ſame thing, the object-glaſs is not diminiſhed in ſo great a proportion as by this preſent one; being not reſtrained by the given ratios of the intervals of the points B, O, P, N.

PROPOSITION IX.

Fig. 553, 554.

715. *If it be required to compoſe a microſcope of two convex lenſes* e *and* p, *which with the given eye-glaſs* e *ſhall magnify in a given ratio, and in which the apparent brightneſs of the object, and the angle of aberration by colours, ſhall be the ſame as in another given microſcope compoſed of two lenſes* E *and* P; *the focal diſtance of the object-glaſs* p *and its aperture and poſition may be found in this manner.*

Suppoſing the ſame ſchemes as before; let the given dimenſions of the old microſcope be theſe; $PD = A$, $PO = C$, $NE = D$, $BO : BP :: 1 : m$. And ſuppoſe the correſponding dimenſions of the new microſcope to be theſe; $pd = a$, $po = c$, $ne = d$, $bo : bq :: 1 : n$; and let the apparent magnitude propoſed, be to the apparent magnitude ſeen by the naked eye from a given diſtance ω, in a given ratio of ω to q. Then we ſhall have $c = \overline{\frac{md}{D} \times \frac{q}{d+q}}\Big|^2 \times C$; $n = \frac{md}{D}\sqrt{\frac{C}{c}}$; and $a = A\sqrt{\frac{c}{C}}$.

For ſince the apparent brightneſs muſt be the ſame in both microſcopes, we muſt put $\frac{PD \times NE}{PN} = \frac{pd \times ne}{pn}$*; that is $\frac{AD}{mC} = \frac{ad}{nc} = f$, to abridge the following reduction. And ſince the angle of aberration by colours muſt alſo be

* Art. 706.

* Art. 708.

* Art. 707.

* Art. 707.

be the ſame in both, we muſt put $\frac{PD}{NE} \times \frac{BP}{BO} = \frac{pd}{ne} \times \frac{bp}{bo}$ *; that is $m\frac{A}{D} =$ * Art. 708.
$n\frac{a}{d} = g$. By the former of theſe equations we have $\frac{ncf}{d} = a = \frac{dg}{n}$ by the latter. Hence $nn = \frac{dd}{c} \times \frac{g}{f} = \frac{dd}{c} \times \frac{mm}{DD}C$, and $n = m\frac{d}{D}\sqrt{\frac{C}{c}}$; and by putting this value for n, we have $a = A\sqrt{\frac{c}{C}}$. Now $q = \frac{ob}{op}ne$ * $= \frac{d}{n-1}$; and * Art. 706.
thence $\frac{d+q}{q} = n = m\frac{d}{D}\sqrt{\frac{C}{c}}$; and $\overline{\frac{d+q}{q}}\Big|^{2} = \overline{\frac{md}{D}}\Big|^{2} \times \frac{C}{c}$; and $c = \overline{\frac{md}{D}}\Big|^{2} \times \overline{\frac{q}{d+q}}\Big|^{2} C$.

716. *Corol.* 1. Hence in theſe microſcopes the diameters of the apertures of the object-glaſſes are in a ſubduplicate ratio of their focal diſtances, as in common teleſcopes. For we had $\frac{a}{A} = \sqrt{\frac{c}{C}}$. And when theſe microſcopes are changed into teleſcopes by making BO and bo infinite, and conſequently $m = 1$ and $n = 1$, we have alſo $\frac{d}{D} = \sqrt{\frac{c}{C}} = \frac{a}{A}$; which agrees with art. 354.

717. *Corol.* 2. In theſe microſcopes the angles of aberration by the figure of the object-glaſſes, are reciprocally as their focal diſtances; and by conſequence are reciprocally in a duplicate ratio of their apertures[a]. [a] Art. 716.
For the angle of aberration by the figure is as $\frac{PD^3}{PO^3} \times \frac{PN}{NE}$ * $= \frac{mA^3}{CCD}$; that is * Art. 709.
this angle in the old microſcope, is to the like angle in the new as $\frac{mA^3}{CCD}$ to $\frac{na^3}{ccd}$, or as $\frac{AA}{CC}$ to $\frac{aa}{cc}$ (becauſe we had $\frac{mA}{D} = \frac{na}{d}$ *,) that is, as $\frac{1}{C}$ to $\frac{1}{c}$; be- * Art. 708.
cauſe $AA : aa :: C : c$ *. * Art. 716.

718. *Corol.* 3. By keeping the ſame object-glaſs and by altering the focal diſtance of the eye-glaſs, the apparent magnitude of the object can be increaſed but little. For by the value of $q = \frac{d}{n-1}$ *, we have $q : d :: 1$ * Art. 706.
$: n-1$ and $q + d : d :: n : n-1$; we have alſo $pb : po :: n : n-1$; therefore $pb = \frac{d+q}{d}c$, and $pb^2 = \overline{\frac{d+q}{d}}\Big|^{2} \times c \times (c)\frac{mm}{DD} \times \overline{\frac{dq}{d+q}}\Big|^{2} \times C = \frac{mm}{DD}cCqq$; and conſequently when c is given, we have pb as q. But pb muſt always be ſomewhat bigger than po; and therefore if at firſt it be put leſs than $2po$, as it muſt be, to make the image bigger than the object; it is manifeſt that pb and conſequently q cannot be diminiſhed ſo much as in the ratio of 2 to 1, that is the apparent magnitude cannot be doubled.

719. *Corol.* 4. But if we keep the ſame eye-glaſs and diminiſh the focal diſtance of the object-glaſs, we might magnify as much as we pleaſe but for the increaſe of the aberrations by the figure[b]. For ſince the value [b] Art. 717.

of po may be thus expreſſed, $c = \overline{\frac{m}{D} \times \frac{d}{\frac{d}{q}+1}}\Big|^{2} \times C$; by increaſing the magnifying power, that is by diminiſhing q, the denominator of this quantity will be increaſed and c will be diminiſhed. Therefore to improve the given microſcope as far as poſſible, we muſt retain much the ſame eye-glaſs and diminiſh the object-glaſs and its aperture according to the rules above, till we find the aberrations by its figure begin to be troubleſome; and if it be required to magnify ſtill more, we may from the laſt microſcope, in which the aberrations by the figure do no harm, determine another by the following propoſition.

PROPOSITION X.

Fig. 553, 554. 720. *If it be required to compoſe a microſcope of two convex lenſes* e *and* p, *which with the given eye-glaſs* e *ſhall magnify in a given ratio; and in which the apparent brightneſs of the object, and the angle of aberration by the figure ſhall be the ſame as in another given microſcope compoſed of two lenſes* E *and* P, *the focal diſtance of the object glaſs* p *and its aperture and poſition may be found in this manner.*

Suppoſing the ſame ſchemes and notation as before, we ſhall have $c = \overline{\frac{md}{D} \times \frac{q}{d+q}}\Big|^{4} C$; $n = \frac{md}{D}\sqrt[4]{\frac{C}{c}}$; $a = A\sqrt[4]{\frac{c^3}{C^3}}$.

For ſince the apparent brightneſs muſt be the ſame in both micro-
* Art. 707. ſcopes, we muſt put $\frac{PD \times NE}{PN} = \frac{pd \times ne}{pn}$*; that is $\frac{AD}{mC} = \frac{ad}{nc} = f$ as before.
And ſince the angle of aberration by the figure muſt be the ſame in both,
* Art. 709. we muſt put $\frac{PD^3}{PO^3} \times \frac{PN}{NE} = \frac{pd^3}{po^3} \times \frac{pn}{ne}$*; that is $\frac{mA^3}{CCD} = \frac{na^3}{ccd} = b$ to ſhorten
the work. Hence we have $a = \frac{ncf}{d}$; and $\frac{n^3c^3f^3}{d^3} = (a^3 =) \frac{ccdb}{n}$; that is $n^4 = \frac{d^4}{c} \times \frac{b}{f^3} = \frac{d^4}{c} \times \frac{m^4}{D^4} C$; and $n = \frac{md}{D}\sqrt[4]{\frac{C}{c}}$. And thence $a = (\frac{ncf}{d} =) A$
* Art. 706. $\sqrt[4]{\frac{C^3}{C^3}}$. But $q = \frac{ob}{op} ne$* $= \frac{d}{n-1}$; and thence $\frac{d+q}{q} = (n =) \frac{md}{D}\sqrt[4]{\frac{C}{c}}$; and $\overline{\frac{d+q}{q}}\Big|^{4} = \overline{\frac{md}{D}}\Big|^{4} \times \frac{C}{c}$; and $c = \overline{\frac{md}{D} \times \frac{q}{d+q}}\Big|^{4} C$.

721. *Corol.* 1. In theſe microſcopes the diameters of the apertures of the object-glaſſes, are as the biquadrate roots of the cubes of their focal diſtances. For we had $\frac{a}{A} = \sqrt[4]{\frac{c^3}{C^3}}$.

722. *Corol.* 2. And the angles of aberration by colours, cauſed by the object-glaſſes, are directly in a ſubduplicate ratio of their focal diſtances.
* Art. 717. The manner of proof is the ſame as before*.

723. *Corol.* 3. Here alſo the magnifying power can be but little increaſed by changing the eye-glaſs. For we ſhall find pb as q, in like manner as before[a].

a Art. 718.

724. *Corol.* 4. In the former propoſition we had po as $\overline{\frac{d}{\frac{d}{q}+1}}\Big|^{2}$, and by the preſent propoſition po is as $\overline{\frac{d}{\frac{d}{q}+1}}\Big|^{4}$; ſo that for the ſame increaſe of the magnifying power, or decreaſe of q, the preſent po decreaſes in a duplicate ratio of the former. For which reaſon it is beſt to magnify as much as we can by the former propoſition before we apply the latter; that the object-glaſs may be preſerved as large as poſſible. And by this microſcope we might magnify to infinity but for the ſmallneſs of the object-glaſs; becauſe the brightneſs will continue the ſame and the diſtinctneſs will be increaſed by cor. 2.

725. *Corol* 5. If a new microſcope be required in which the angles of aberration of both kinds ſhall be reſpectively the ſame as in the old one; we muſt retain the ſame object-glaſs; and to magnify more we muſt increaſe the focal diſtance of the eye-glaſs; but if it be increaſed to infinity the apparent magnitude in the new microſcope, will be to the apparent magnitude in the old, but as m to $m-1$, or in *Huygens*'s ſtandard but as 10 to 9, which is but a trifle. For the angle of aberration by the figure will be the ſame by putting $\frac{mA^3}{CCD}=\frac{na^3}{ccd}$*, in which by ſubſtituting the values of n and a found in the former propoſition, which ſuppoſed the angle of aberration by colours to be the ſame, we ſhall find $c=C$; therefore in the value of $c=\overline{\frac{md}{D}\times\frac{q}{d+q}}\Big|^{2}C$*; we muſt put $\frac{md}{D}\times\frac{q}{d+q}=1$; which gives $d=\frac{1}{\frac{m}{D}-\frac{1}{q}}$; by which it appears that d will increaſe by diminiſhing q; and will become infinite when $\frac{m}{D}-\frac{1}{q}=0$, or when $q=\frac{D}{m}$; but in the old microſcope $Q=\frac{D}{m-1}$*; therefore we have Q to q, or the apparent magnitude in the new, to the apparent magnitude in the old, as m to $m-1$. It is to no purpoſe therefore to alter the eye-glaſs as we ſhewed before.

* Art. 709.

* Art. 715.

* Art. 706.

726. To give an example of each of theſe propoſitions; in *Huygens*'s ſtandard microſcope we have $NE=D=2$; $PD=A=\frac{1}{20}$; $PO=C=\frac{1}{10}$; $PB=\frac{1}{9}$; whence $m=10$ and $Q=\frac{D}{m-1}$*$=\frac{2}{9}$. To magnify as much

* Art. 706.

* Art. 706. more, we muſt put $q = \frac{1}{9}$*. Hence by the former propoſition putting $d = D = 2$ we have $po = c = \frac{70}{361}$; $n = 19$; $pb = \frac{35}{171}$; and $pd = a = \frac{1}{38}$: But by this propoſition $po = c = \frac{7000}{191^{+}} = \frac{1}{19}$ nearly, and the reſt according to the rules. If we put $d = 1$, by the former propoſition we ſhall have $po = \frac{7}{40}$; and by the preſent propoſition $po = \frac{7}{160}$; which agree with the rules in the 710 and 713 articles.

LEMMA VII.

727. Concerning the apparent magnitude, brightneſs and diſtinctneſs of objects in reflecting microſcopes, compoſed of a reflecting concave-metal and a ſingle eye-glaſs.

Fig. 555. Let an object BX be placed between the center P and principal focus $\mathcal{T}$ of a reflecting concave ſurface ACG; and let the image NY be viewed through the eye-glaſs EZ, whoſe focal diſtance is NE. Let the object BX be viewed alſo by the naked eye from any given diſtance $B\Omega$; then take $B\mathcal{Q}$ to $B\mathcal{T}$ as NE to $\mathcal{T}C$, and the object will be magnified in the microſcope in the ratio of $B\Omega$ to $B\mathcal{Q}$.

For any line PX, produced both ways to the concave at G and to the eye-glaſs at Z, is the axis of an inclined pencil of rays, flowing from X, reflected from G to the image at Y, and refracted at Z to the eye at V.
* Art. 50. Draw $X\mathcal{Q}$ parallel to ZV or EY*, and the object will appear in the microſcope under the angle EVZ equal to NEY or $B\mathcal{Q}X$; and conſequently will be magnified in the ratio of the angle $B\mathcal{Q}X$ to $B\Omega X$ or of $B\Omega$ to $B\mathcal{Q}$. But $B\mathcal{Q} : NE :: (BX : NY :: PB : PN ::)\ \mathcal{T}B : \mathcal{T}P$ or $\mathcal{T}C$; be-
a Art. 207. becauſe $\mathcal{T}B$, $\mathcal{T}P$, $\mathcal{T}N$ are continual proportionals[a]. $\mathcal{Q}$. E. D.

728. *Corol.* 1. The apparent diſtance $B\mathcal{Q} = \frac{\mathcal{T}B}{\mathcal{T}C} \times NE = \frac{PB}{PN} \times NE = \frac{CB}{CN} \times NE$: and $B\Omega$ being given, the apparent magnitude of the object is reciprocally as $B\mathcal{Q}$, by the lemma.

729. *Corol.* 2. Let CA be the ſemiaperture of the reflecting concave, and the apparent brightneſs of the ſame object, in the ſame or in different microſcopes, will be as $\frac{CA^2 \times NE^2}{CN^2}$. For it is directly as the quantity of light received by the ſurface of the concave from any particle B, and inverſely as the area of the picture of that particle upon the retina or inverſely as the apparent magnitude of the area of the particle; and conſequently is as $\frac{CA^2}{CB^2}$ * $\times \frac{CB^2 \times NE^2}{CN^2}$ *.

* Art. 58.
* Art. 728.

730. *Corol.* 3. The apparent indiſtinctneſs of a given object, in the ſame or in different reflecting microſcopes, is as $\frac{CA^6}{CT^6}\times\frac{PN^4}{CN^2\times NE^2}$; and conſequently is invariable when $\frac{CA^3}{CT^3}\times\frac{PN^2}{CN\times NE}$ is invariable. For the greateſt lateral aberration in the picture at *N* is as $\frac{CA^3}{CT^3}\times\frac{PN^2}{NE}$ * and the apparent indiſtinctneſs is as the ſquare of this aberration directly and as NE^2 inverſely [a], neglecting the aberrations cauſed by the eye-glaſs according to art. 702.

* Art. 702.

[a] Art. 696.

PROPOSITION XI.

731. *A new reflecting microſcope denoted by ſmall letters* abcpnev, *may be made to magnify more than a given reflecting microſcope, adjuſted by experiments and denoted by larger letters* ABCPNEV, *in any given ratio of* n *to* 1, *with the ſame brightneſs and diſtinctneſs too pretty nearly, by taking* $\mathrm{ne}=\frac{1}{\mathrm{n}}\mathrm{NE}$; $\mathrm{ca}=\frac{1}{\mathrm{n}^3}\mathrm{CA}$; $\mathrm{ct}=\frac{1}{\mathrm{n}^4}\mathrm{CT}$; $\mathrm{cb}=\frac{1}{\mathrm{n}^4}\mathrm{CB}$; $\mathrm{cn}=\frac{1}{\mathrm{n}^4}\mathrm{CN}$. Fig. 555, 556.

732. This propoſition may be demonſtrated by the foregoing lemma and its corollaries, in the ſame manner as the 8th propoſition was demonſtrated by its lemma. For here alſo the intervals of the points *C*, *B*, *P*, *N* are ſuppoſed to be in given ratios. But the following propoſition is more general and better than this, becauſe it will not diminiſh the focal diſtance of the concave ſo much as this does.

PROPOSITION XII.

733. *Having a reflecting microſcope conſiſting of a concave metal* CA *and a convex eye-glaſs* EZ *adjuſted together by experiment; it is propoſed to adjuſt any other given concave* ca *and convex eye-glaſs* ez, *ſo that the apparent brightneſs of the object ſhall continue the ſame as in the given microſcope, and the apparent diſtinctneſs too, neglecting the increment of the aberrations cauſed by the new eye-glaſs* ez *if it be taken leſs than* EZ: *and then to ſhew how much the new microſcope will magnify.* Fig. 555, 556.

In the 555th figure already deſcribed, let the dimenſions of the given microſcope be theſe; $CA=A$, $CT=C$, $NE=D$ and $TB:TC::1:m$; and in the new microſcope, whoſe correſponding parts are denoted by ſmall letters of the ſame names, let the given lines $tc=c$, $ne=d$. Take a number $n=\sqrt{1+\overline{mm-1},\frac{dd}{DD}\sqrt{\frac{C}{c}}}$; and for the places of the object and of the eye-glaſs we ſhall have $tb:tc$ and $tc:tn::1:n$; and the

ſemiaperture ca or $a = \frac{n+1}{m+1} \times \frac{cD}{Cd} A$: and the apparent magnitude of the object in the new microſcope, will be to its apparent magnitude, ſeen by
* Art. 728. the naked eye, from a given diſtance ω, as ω to $\frac{d}{n}$* or $\frac{d}{\sqrt{1+\overline{mm-1},\frac{dd}{DD}\sqrt{\frac{C}{c}}}}$.

For ſince the apparent brightneſs muſt be the ſame in both micro-
* Art. 729. ſcopes, we muſt put $\frac{CA \times NE}{CN} = \frac{ca \times ne}{cn}$*, that is $\frac{AD}{\overline{m+1},C} = \frac{ad}{\overline{n+1},c} = f$, to ſhorten the following reduction. And ſince the apparent diſtinctneſs muſt
* Art. 730. be the ſame too, we muſt put $\frac{CA^3}{CT^3} \times \frac{PN^2}{CN \times NE} = \frac{ca^3}{ct^3} \times \frac{pn^2}{cn \times ne}$*; that is $\frac{\overline{m-1}^2}{m+1} \times \frac{A^3}{CCD} = \frac{\overline{n-1}^2}{n+1} \times \frac{a^3}{ccd} = g$. By the former equation we have $a = \frac{\overline{n+1},cf}{d}$, and by cubing $\frac{\overline{n+1}^3,c^3f^3}{d^3} = (a^3 =) \frac{n+1}{\overline{n-1}^2} \times ccdg$, by the latter equation. Hence $\overline{nn-1}^2 = \frac{d^4g}{cf^3}$; and $nn-1 = dd\sqrt{\frac{g}{cf^3}} = dd \times \frac{mm-1}{DD} \times \sqrt{\frac{C}{c}}$, by reſtoring the values of f and g; and $n = \sqrt{1+\overline{mm-1}\frac{dd}{DD}\sqrt{\frac{C}{c}}}$. And by ſubſtituting the value of f in this equation $a = \frac{\overline{n+1},cf}{d}$, we have the value of a as above. Since the breadth of the middle pencil, where it
a Art. 729. 714. emerges from any eye-glaſs, is always the ſame[a]; if we retain the ſame eye-glaſs, it will cauſe the ſame aberrations; but if we diminiſh the eye-glaſs, by a greater refraction it will increaſe its own aberrations and conſequent-
b Art. 702. Fig. 552. ly the apparent indiſtinctneſs. For though the bigger part[b] of the whole angle of aberration aib or Rir is kept invariable; yet the leſſer part of it, which is ſubtended by qr, will vary reciprocally as qe, taking qr for the aberration by colours; and reciprocally as qe cube, taking qr for the aberration cauſed by the figure of the eye-glaſs. For this part of that angle
* Art. 222. being as $\frac{qr}{qe}$*, is as $\frac{ei}{eq}$ in the firſt caſe[c], and as $\frac{ei^3}{eq^3}$ in the ſecond caſe[d],
c Art. 327. d Art. 700. ſuppoſing we always apply ſimilar eye-glaſſes. Q. E. D.

734. *Corol.* 1. Hence when ω is given, the apparent magnitude in this microſcope is as $\frac{n}{d}$ or as $\sqrt{\frac{1}{dd}+\frac{mm-1}{DD}\sqrt{\frac{C}{c}}}$; and conſequently may be increaſed at pleaſure by diminiſhing c; and alſo by diminiſhing d, if we neglect the ſmall increment of the aberrations cauſed by ſmaller eye-glaſſes.

735. *Corol.* 2. Hence putting $q = \frac{d}{n}$ the apparent diſtance of the object[a]; if c and q be given and the reſt be required, we ſhall have $d = \frac{qD}{\sqrt{DD + \overline{1-mm},qq\sqrt{\frac{C}{c}}}}$; and conſequently $n = \frac{d}{q}$, and $tb : tc :: 1 : n$, and $a = \frac{n+1}{m+1} \times \frac{cD}{Cd} A$. For ſince $q = (\frac{d}{n} =) \frac{d}{\sqrt{1 + \overline{mm-1}, \frac{dd}{DD}\sqrt{\frac{C}{c}}}}$, by reduction we ſhall find d as above. If the old concave be retained we ſhall have $d = \frac{D}{\sqrt{\frac{DD}{qq} + 1 - mm}}$.

736. *Corol.* 3. If d and q be given and the reſt be required, we ſhall have $c = \overline{\frac{mm-1}{DD} \times \frac{ddqq}{dd-qq}}\Big|^{2} \times C$, by reduction as before; and tb and ca the ſame as in corol. 2. If the old eye-glaſs be retained, we ſhall have $c = \overline{\overline{mm-1} \times \frac{aq}{DD-qq}}\Big|^{2} \times C$, and the apparent diſtinctneſs will continue the very ſame as in the given microſcope.

737. *Corol.* 4. This microſcope may be changed into Sir *Iſaac Newton*'s reflecting teleſcope by making TB and tb infinite, and conſequently $m = 0$ and $n = 0$, that is $1 - \frac{dd}{DD}\sqrt{\frac{C}{c}} = 0$. Hence $\frac{D}{d} = \sqrt[4]{\frac{C}{c}}$ or $d : D :: \sqrt[4]{c} : \sqrt[4]{C}$. We have alſo $\frac{a}{A} = \frac{cD}{Cd} = \frac{c}{C}\sqrt[4]{\frac{C}{c}} = \sqrt[4]{\frac{c^3}{C^3}}$ that is $a : A :: \sqrt[4]{c^3} : \sqrt[4]{C^3}$. And becauſe $\frac{C}{D} a = \frac{c}{d} A$, we have $\frac{c}{d} : \frac{C}{D} :: (a : A ::) \sqrt[4]{c^3} : \sqrt[4]{C^3}$, or the magnifying powers as $\sqrt[4]{C^3}$. All which agree with art. 361.

Dimenſions of a reflecting microſcope.

738. Having made a few groſs experiments with a concave metal that I had by me, whoſe focal diſtance was $\frac{2}{3}$ inch, and with ſeveral convex eye-glaſſes applied to it; I found that the colours of objects in a reflecting microſcope appeared much more beautiful and natural than in double refracting microſcopes of the beſt ſort; their proper colours being free from the mixture of other colours ariſing in refracting microſcopes from the different refrangibility of rays. When ſome ſmall hairs and a mite were placed upon a plane piece of glaſs at B, and had ſcarce any other illumination than the direct ſky-light at a window, I found they appeared ſufficiently bright and very diſtinct when the dimenſions of my reflecting microſcope were theſe in parts of an inch; CA or $A = \frac{1}{2}$; CT or $C = \frac{2}{3}$; NE or $D = \frac{2}{3}$; $TB : TC :: 1 : 14$. Therefore we have $m = 14$, and the

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[a] Art. 728.

apparent diſtance $B\mathcal{Q} = \frac{D}{m}$[*] $= \frac{1}{6}$; and conſequently, putting $B\Omega = 8$ inches, theſe objects were magnified 48 times in diameter.

739. Hence it is eaſy to compute from corol. 3, that to magnify 72 times with the ſame eye-glaſs of $\frac{7}{3}$ inch focal diſtance, and conſequently with the very ſame diſtinctneſs and brightneſs as in this ſtandard experiment, the focal diſtance ct of the new concave muſt be 0, 458 of an inch; and conſequently the diameter of the ſpherical ſurface of which this concave is a portion, that is $4\,ct$, is 1, 832. Which is above 9 times bigger than $p\rho = \frac{70}{361} = 0,194$, the focal diſtance and diameter of the ſphere of which that plano-convex lens is a portion, which we found would alſo magnify 72 times with an eye-glaſs of 2 inches focal diſtance[a]; but not ſo diſtinctly as its ſtandard did; becauſe the angle of aberration by the ſpherical figure of the object-glaſs was increaſed almoſt 4 times[b]. Now this exceſs of the diameter of the ſphere of the concave metal above that of the lens, is a conſiderable advantage in order to magnify ſtill farther by a farminution of the concave.

740. But how far the magnifying power of a given concave may be ſucceſsfully promoted by diminiſhing the eye-glaſs according to the rule in corol. 2, I could not examine experimentally for want of proper mechaniſm, to adjuſt the intervals between the concave, the object and the eye-glaſs, according to computation; and alſo to hold a convex lens or a reflecting concave in a proper poſition for caſting light upon the object; and to compute it by theory would be troubleſome.

741. Theſe I ſuppoſe may be the advantages which Sir *Iſaac Newton* expected, when he tells us he ſometimes had thoughts of making a microſcope of this ſort[c]. Nevertheleſs in attempting to magnify very much we ſhall ſoon be ſtopt here too, by the minuteneſs of the concaves that are neceſſary for the purpoſe. This put me upon contriving a microſcope with two reflecting ſpherical ſurfaces of any ſize, ſo proportioned to each other that the aberrations of the rays cauſed by the firſt reflection, ſhall be perfectly corrected by the ſecond; and by conſequence that the laſt image of the object, from which the rays diverge upon the eye-glaſs, ſhall be as perfectly free from aberrations, as the theory of theſe aberrations is perfect. But having made the demonſtration of this conſtruction quite independent upon any thing contained in this troubleſome chapter, I chuſe to give it as a Remark.

A COM-

[*] Art. 728.

[a] Art. 726.

[b] Art. 717.

[c] Phil. Tranſ. N°. 80.